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Unbounded subnormal composition operators in L^2-spaces ☆

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A novel criterion for subnormality of unbounded composition operators in L^2-spaces, written in terms of measurable families of probability measures satisfying the so-called consistency condition, is established. It becomes a new characterization of subnormality in the case of bounded composition operators. Pseudo-moments of a measurable family of probability measures that satisfies the consistency condition are proved to be given by the Radon–Nikodym derivatives which appear in Lambert's characterization of bounded composition operators. A criterion for subnormality of composition operators induced by matrices is provided. The question of subnormality of composition operators over discrete measure spaces is studied. Two new classes of subnormal composition operators over discrete measure spaces are introduced. A recent criterion for subnormality of weighted shifts on directed trees by the present authors is essentially improved in the case of rootless

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directed trees and nonzero weights by dropping the assumption of density of C^∞-vectors in the underlying ℓ^2-space.

1. Preliminaries

1.1. Introduction

In 1950 Halmos introduced the notion of a bounded subnormal operator and gave its first characterization (cf. [34]), which was successively simplified by Bram [8], Embry [27] and Lambert [40]. Neither of them is true for unbounded operators (see [22] and [61–63] for foundations of the theory of bounded and unbounded subnormal operators). The only known general characterizations of subnormality of unbounded operators refer to semispectral measures or elementary spectral measures (cf. [7,31,68]). They seem to be useless in the context of particular classes of operators. The other known criteria for subnormality (with the exception of [69]) require the operator in question to have an invariant domain (cf. [62,65,21,2]). In this paper we give a criterion for subnormality of densely defined composition operators (in L^2-spaces) with no additional restrictions.

Composition operators occur in many areas of mathematics. They play a vital role in ergodic theory and functional analysis. The theory of bounded composition operators seems to be well-developed (see [53,46,71,36,41,42,26,30,55,17,15,16]; see also [28,43,56, 25,59] for particular classes of such operators). As opposed to the bounded case, the theory of unbounded composition operators is at a rather early stage of development. There are few papers concerning this issue. Some basic facts about unbounded composition operators can be found in [18,37,13,10]. To the best of our knowledge, there is no paper concerning the question of subnormality of (general) unbounded composition operators. A criterion for subnormality of certain composition operators built over directed trees can be deduced from [11, Theorem 5.1.1] via [39, Lemma 4.3.1]. However, it requires the operator in question to have dense set of C^∞-vectors. The reason for this is that its proof is based on an approximation technique derived from [21, Theorem 21] in which the invariance of the domain plays an essential role. In other words, this technique could not be applied when looking for a general criterion for subnormality of unbounded composition operators. On the other hand, Lambert's characterization of bounded subnormal composition operators, which is written in terms of the Radon–Nikodym derivatives $\{\mathsf{h}_{\phi^n}\}_{n=0}^\infty$ (cf. (3)), is no longer valid in the unbounded case (see [39, Theorem 4.3.3] and [13, Section 11]).

In the present paper we give the first ever criterion for subnormality of unbounded composition operators, which becomes a new characterization of subnormality in the bounded case. It states that if an injective densely defined composition operator has a measurable family of probability measures that satisfies the so-called consistency condition, then it is subnormal (cf. Theorem 9). The consistency condition appeals to the

Radon–Nikodym derivative h_ϕ. To invent it, we revisit Lambert's construction of a quasinormal extension of a bounded subnormal composition operator which is given in [42]. Surprisingly, the pseudo-moments of a measurable family of probability measures that satisfies the consistency condition are given by the Radon–Nikodym derivatives $\{\mathsf{h}_{\phi^n}\}_{n=0}^\infty$ (cf. Theorem 17).

The paper consists of three parts. The first contains some background material concerning Stieltjes moment sequences, composition operators and conditional expectation (with respect to $\phi^{-1}(\mathscr{A})$). The second consists of four sections. Section 2.1 provides the main criterion for subnormality of unbounded composition operators (cf. Theorem 9). That this criterion becomes a characterization in the bounded case is justified in Section 2.2. The consistency condition is investigated in Section 2.3. In particular, it is proved that the consistency condition behaves well with respect to the operation of taking powers of composition operators (cf. Proposition 23). Section 2.4 deals with the strong consistency condition, a variant of the consistency condition which does not appeal to conditional expectation. It is shown that in the bounded case the strong consistency condition is equivalent to requiring that the Radon–Nikodym derivatives $\{\mathsf{h}_{\phi^n}\}_{n=0}^\infty$ be invariant for the operator of conditional expectation (cf. Proposition 30). The third part of the paper deals with particular classes of bounded or unbounded composition operators. In Section 3.1 we prove that composition operators in $L^2(\mu_\gamma)$ induced by normal $\kappa \times \kappa$ matrices are subnormal, where μ_γ is a Borel measure on $\mathbb{R}^\kappa$ with a density function given by an entire function with nonnegative Taylor coefficients at 0 (cf. Theorem 32). The question of subnormality of composition operators in L^2-spaces over discrete measure spaces is reexamined in Section 3.2 (cf. Theorem 35). A model for such operators with injective symbols is established in Remark 37. In Section 3.3 we introduce a "local consistency technique" which is new even in the bounded case (cf. Lemma 38). It enables us to deduce subnormality of a composition operator in an L^2-space over a discrete measure space from the Stieltjes determinacy of the Radon–Nikodym derivatives $\{\mathsf{h}_{\phi^{n+1}}\}_{n=0}^\infty$ (cf. Theorem 41). In Section 3.4 we use the "local consistency technique" to model subnormal composition operators induced by a transformation which has only one essential fixed point. Section 3.5 deals with the question of subnormality of a class of composition operators over a directed tree with finite constant valences on generations. In this case, even though the operator of conditional expectation is far from being the identity, we can use the strong consistency condition. This enables us to characterize subnormality within this class by using Lambert's condition (cf. Theorem 44), the phenomenon known so far for unilateral and bilateral injective weighted shifts only. In Section 3.6 we show that Theorem 5.1.1 of [11], which is a criterion for subnormality of a weighted shift on a directed tree, remains valid if the assumption that C^∞-vectors are dense is dropped, provided the weights are nonzero and the tree is rootless and leafless (cf. Theorem 47).

The paper is concluded with Appendices A, B and C concerning composition operators induced by roots of the identity, symmetric composition operators and orthogonal sums of composition operators.

1.2. Prerequisites

We write $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ for the sets of integers, real numbers and complex numbers, respectively. We denote by $\mathbb{N}$, $\mathbb{Z}_+$ and $\mathbb{R}_+$ the sets of positive integers, nonnegative integers and nonnegative real numbers, respectively. Set $\overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{\infty\}$. In what follows, we adhere to the convention that $0 \cdot \infty = \infty \cdot 0 = 0$, $\frac{1}{0} = \infty$ and $\frac{0}{0} = 1$. If $\zeta\colon X \to \overline{\mathbb{R}}_+$ is a function on a set X, then we put $\{\zeta = 0\} = \{x \in X\colon \zeta(x) = 0\}$ and $\{\zeta > 0\} = \{x \in X\colon \zeta(x) > 0\}$. Given subsets $\varDelta, \varDelta_n$ of X, $n \in \mathbb{N}$, we write $\varDelta_n \nearrow \varDelta$ as $n \to \infty$ if $\varDelta_n \subseteq \varDelta_{n+1}$ for every $n \in \mathbb{N}$ and $\varDelta = \bigcup_{n=1}^\infty \varDelta_n$. The characteristic function of a subset $\varDelta$ of X is denoted by $\chi_\varDelta$. The symbol $\sigma(\mathscr{P})$ is reserved for the σ-algebra generated by a family $\mathscr{P}$ of subsets of X. All measures considered in this paper are assumed to be positive. Given two measures μ and ν on the same σ-algebra, we write $\mu \ll \nu$ if μ is absolutely continuous with respect to ν; then $\frac{\mathrm{d}\mu}{\mathrm{d}\nu}$ stands for the Radon–Nikodym derivative of μ with respect to ν (provided it exists). We shall abbreviate the expressions "almost everywhere with respect to μ" and "for μ-almost every x" to "a.e. $[\mu]$" and "for μ-a.e. x", respectively. As usual, $L^2(\mu)$ stands for the Hilbert space of all square integrable (with respect to a measure μ) complex functions on X. If μ is the counting measure on X, then we write $\ell^2(X)$ in place of $L^2(\mu)$. The σ-algebra of all Borel sets of a topological space Z is denoted by $\mathfrak{B}(Z)$. In what follows δ_t stands for the Borel probability measure on $\mathbb{R}_+$ concentrated at $t \in \mathbb{R}_+$. The closed support of a finite Borel measure ν on $\mathbb{R}_+$ is denoted by $\operatorname{supp}\nu$.

Now we state an auxiliary lemma which follows from [45, Proposition I-6-1] and [3, Theorem 1.3.10].

Lemma 1. *Let $\mathscr{P}$ be a semi-algebra of subsets of a set X and μ_1, μ_2 be measures on $\sigma(\mathscr{P})$ such that $\mu_1(\varDelta) = \mu_2(\varDelta)$ for all $\varDelta \in \mathscr{P}$. Suppose there exists a sequence $\{\varDelta_n\}_{n=1}^\infty \subseteq \mathscr{P}$ such that $\varDelta_n \nearrow X$ as $n \to \infty$ and $\mu_1(\varDelta_k) < \infty$ for every $k \in \mathbb{N}$. Then $\mu_1 = \mu_2$.*

From now on, we write $\int_0^\infty$ instead of $\int_{\mathbb{R}_+}$. A sequence $\{a_n\}_{n=0}^\infty \subseteq \mathbb{R}$ is said to be a *Stieltjes moment sequence* if there exists a Borel measure ν on $\mathbb{R}_+$, called a *representing measure* of $\{a_n\}_{n=0}^\infty$, such that

$$a_n = \int_0^\infty s^n \nu(\mathrm{d}s), \quad n \in \mathbb{Z}_+.$$

If such a ν is unique, then $\{a_n\}_{n=0}^\infty$ is called *determinate*. A Borel measure ν on $\mathbb{R}_+$ is said to be *determinate* if all its moments $\int_0^\infty s^n \nu(\mathrm{d}s)$, $n \in \mathbb{Z}_+$, are finite and the Stieltjes moment sequence $\{\int_0^\infty s^n \nu(\mathrm{d}s)\}_{n=0}^\infty$ is determinate. Sequences or measures which are not determinate are called *indeterminate*. Recall that any finite Borel measure on $\mathbb{R}_+$ with compact support is determinate (cf. [32]). Another criterion for determinacy can be deduced from the M. Riesz theorem (cf. [32] and [39, Lemma 2.2.5]).

$$\text{A Borel measure } \nu \text{ on } \mathbb{R}_+ \text{ whose all moments are finite and } \nu(\{0\}) = 0 \\ \text{is determinate if and only if } \mathbb{C}[t] \text{ is dense in } L^2((1+t^2)\nu(\mathrm{d}t)), \tag{1}$$

where $\mathbb{C}[t]$ stands for the ring of all complex polynomials in real variable t. We refer the reader to [5, Proposition 1.3] for a full characterization of determinacy. The following useful lemma is related to [50, Exercise 23, Chapter 3]. We include its proof to keep the exposition as self-contained as possible.

Lemma 2. *If $\{a_n\}_{n=0}^\infty \subseteq (0,\infty)$ is a Stieltjes moment sequence with a representing measure ν, then the sequence $\{\frac{a_{n+1}}{a_n}\}_{n=0}^\infty$ is monotonically increasing and*

$$\sup_{n\in\mathbb{Z}_+} \frac{a_{n+1}}{a_n} = \sup(\operatorname{supp}\nu).$$

Proof. Applying the Cauchy–Schwarz inequality, we deduce that the sequence $\{\frac{a_{n+1}}{a_n}\}_{n=0}^\infty$ is monotonically increasing. This implies that

$$\sup_{n\in\mathbb{Z}_+} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} \overset{(\dagger)}{=} \lim_{n\to\infty} \sqrt[n]{a_n} \overset{(\ddagger)}{=} \sup(\operatorname{supp}\nu),$$

where ($\dagger$) and ($\ddagger$) may be inferred from [58, Lemma 2.2] (with $\Omega = \mathbb{Z}_+$, $A(n) = n+1$ and $\phi(n) = a_n$) and [50, Exercise 4, Chapter 3], respectively. $\square$

Let A be an operator in a complex Hilbert space $\mathcal{H}$ (all operators considered in this paper are linear). Denote by $\mathcal{D}(A)$, $\mathcal{N}(A)$, $\mathcal{R}(A)$ and A^* the domain, the kernel, the range and the adjoint of A (in case it exists) respectively. Set $\mathcal{D}^\infty(A) = \bigcap_{n=0}^\infty \mathcal{D}(A^n)$ with $A^0 = I$, where $I = I_{\mathcal{H}}$ stands for the identity operator on $\mathcal{H}$. Members of $\mathcal{D}^\infty(A)$ are called *C^∞-vectors* of A. A vector subspace $\mathcal{E}$ of $\mathcal{D}(A)$ is called a *core* for A if $\mathcal{E}$ is dense in $\mathcal{D}(A)$ with respect to the graph norm of A. If A is closed and densely defined, then A has a (unique) *polar decomposition* $A = U|A|$, where U is a partial isometry on $\mathcal{H}$ such that the kernels of U and A coincide and $|A|$ is the square root of A^*A (cf. [6, Section 8.1]). Given two operators A and B in $\mathcal{H}$, we write $A \subseteq B$ if $\mathcal{D}(A) \subseteq \mathcal{D}(B)$ and $Af = Bf$ for all $f \in \mathcal{D}(A)$. In what follows $\boldsymbol{B}(\mathcal{H})$ stands for the C^*-algebra of all bounded operators in $\mathcal{H}$ whose domains are equal to $\mathcal{H}$. A densely defined operator N in $\mathcal{H}$ is said to be *normal* if N is closed and $N^*N = NN^*$ (or equivalently if and only if $\mathcal{D}(N) = \mathcal{D}(N^*)$ and $\|Nf\| = \|N^*f\|$ for all $f \in \mathcal{D}(N)$, see [6]). We say that a densely defined operator S in $\mathcal{H}$ is *subnormal* if there exist a complex Hilbert space $\mathcal{K}$ and a normal operator N in $\mathcal{K}$ such that $\mathcal{H} \subseteq \mathcal{K}$ (isometric embedding), $\mathcal{D}(S) \subseteq \mathcal{D}(N)$ and $Sf = Nf$ for all $f \in \mathcal{D}(S)$. Since powers of a normal operator are normal, we see that any densely defined power of a subnormal operator is still subnormal. The members of the next class are related to subnormal operators. A closed densely defined operator A in $\mathcal{H}$ is said to be *quasinormal* if $U|A| \subseteq |A|U$, where $A = U|A|$ is the polar decomposition of A. Recall that quasinormal operators are subnormal (see [9, Theorem 1] and [62, Theorem 2]). The reverse implication does not hold in general. It is well-known that if S is subnormal, then $\{\|S^n f\|^2\}_{n=0}^{\infty}$ is a Stieltjes moment sequence for every $f \in \mathcal{D}^{\infty}(S)$ (see [11, Proposition 3.2.1]). The converse does not always hold, even if $\mathcal{D}^{\infty}(S)$ is dense in $\mathcal{H}$ (see [11, Section 3.2]).

Let $(X, \mathscr{A}, \mu)$ be a σ-finite measure space. A map from X to X is called a *transformation* of X. Let ϕ be an $\mathscr{A}$*-measurable* transformation of X, i.e., $\phi^{-1}(\varDelta) \in \mathscr{A}$ for all $\varDelta \in \mathscr{A}$. Denote by $\mu \circ \phi^{-1}$ the measure on $\mathscr{A}$ given by $\mu \circ \phi^{-1}(\varDelta) = \mu(\phi^{-1}(\varDelta))$ for $\varDelta \in \mathscr{A}$. We say that ϕ is *nonsingular* if $\mu \circ \phi^{-1}$ is absolutely continuous with respect to μ. The following is easily seen to be true.

$$\begin{gathered}\text{If } \phi \text{ is nonsingular, } Y \text{ is a nonempty set and } f, g\colon X \to Y \text{ are} \\ \text{functions such that } f = g \text{ a.e. } [\mu], \text{ then } f \circ \phi = g \circ \phi \text{ a.e. } [\mu]. \end{gathered} \tag{2}$$

Clearly, if ϕ is nonsingular, then the map $C_\phi\colon L^2(\mu) \supseteq \mathcal{D}(C_\phi) \to L^2(\mu)$ given by

$$\mathcal{D}(C_\phi) = \{f \in L^2(\mu)\colon f \circ \phi \in L^2(\mu)\} \text{ and } C_\phi f = f \circ \phi \text{ for } f \in \mathcal{D}(C_\phi),$$

is well-defined (and linear); the converse is true as well. Such C_ϕ is called a *composition operator* with a *symbol* ϕ (or *induced* by ϕ). Note that every composition operator is closed (cf. [13, Proposition 3.2]). If ϕ is nonsingular, then by the Radon–Nikodym theorem there exists a unique (up to sets of measure μ zero) $\mathscr{A}$-measurable function $\mathsf{h}_\phi\colon X \to \overline{\mathbb{R}}_+$ such that

$$\mu \circ \phi^{-1}(\varDelta) = \int_{\varDelta} \mathsf{h}_\phi \, \mathrm{d}\mu, \quad \varDelta \in \mathscr{A}. \tag{3}$$

Recall that $\mathcal{D}(C_\phi) = L^2(\mu)$ if and only if $\mathsf{h}_\phi \in L^\infty(\mu)$; moreover, if $\mathsf{h}_\phi \in L^\infty(\mu)$, then $C_\phi \in \boldsymbol{B}(L^2(\mu))$ and $\|C_\phi\|^2 = \|\mathsf{h}_\phi\|_{L^\infty(\mu)}$ (see e.g., [46, Theorem 1]). It is well-known that (cf. [18, Lemma 6.1])

$$\text{if } \phi \text{ is nonsingular, then } C_\phi \text{ is densely defined if and only if } \mathsf{h}_\phi < \infty \text{ a.e. } [\mu]. \tag{4}$$

Note also that (cf. [13, Proposition 6.5])

$$\text{if } \phi \text{ is nonsingular, then } \mathsf{h}_\phi \circ \phi > 0 \text{ a.e. } [\mu]. \tag{5}$$

The following fact is patterned on the integral formula due to Embry and Lambert (cf. [29, p. 168]).

Proposition 3. *Let $(X, \mathscr{A}, \mu)$ be a σ-finite measure space and ϕ be a nonsingular transformation of X such that $\mathsf{h}_\phi < \infty$ a.e. $[\mu]$. Then*

$$\int_X \frac{f \circ \phi}{\mathsf{h}_\phi \circ \phi} \, \mathrm{d}\mu = \int_{\{\mathsf{h}_\phi > 0\}} f \, \mathrm{d}\mu \text{ for any } \mathscr{A}\text{-measurable function } f \colon X \to \overline{\mathbb{R}}_+. \tag{6}$$

Proof. Apply (5) and the measure transport theorem (cf. [3, Theorem 1.6.12]) to the restriction of ϕ to a set of μ-full measure on which $\mathsf{h}_\phi \circ \phi$ is positive. $\square$

Given $n \in \mathbb{N}$, we denote by ϕ^n the n-fold composition of ϕ with itself; ϕ^0 is the identity transformation id_X of X. We write $\phi^{-n}(\varDelta) = (\phi^n)^{-1}(\varDelta)$ for $\varDelta \in \mathscr{A}$ and $n \in \mathbb{Z}_+$. If ϕ is nonsingular and $n \in \mathbb{Z}_+$, then ϕ^n is nonsingular and thus h_{ϕ^n} makes sense. It is clear that $\mathsf{h}_{\phi^0} = 1$ a.e. $[\mu]$.

The question of when a (not necessarily densely defined) composition operator is bounded from below has an explicit answer.

Proposition 4. *Let $(X, \mathscr{A}, \mu)$ be a σ-finite measure space and ϕ be a nonsingular transformation of X. If c is a positive real number, then the following two conditions are equivalent:*

(i) $\|C_\phi f\| \geqslant c\|f\|$ *for every* $f \in \mathcal{D}(C_\phi)$,
(ii) $\mathsf{h}_\phi \geqslant c^2$ *a.e.* $[\mu]$.

Proof. If (i) holds, then

$$\int_X (\mathsf{h}_\phi - c^2)|f|^2 \, \mathrm{d}\mu \geqslant 0, \quad f \in \mathcal{D}(C_\phi). \tag{7}$$

Since μ is σ-finite, there exists a sequence $\{X_n\}_{n=1}^\infty \subseteq \mathscr{A}$ such that $\mu(X_k) < \infty$ for every $k \geqslant 1$, and $X_n \nearrow X$ as $n \to \infty$. Set $Y_n = X_n \cap \{x \in X \colon \mathsf{h}_\phi \leqslant n\}$ for $n \geqslant 1$. Fix $n \geqslant 1$. It is easily seen that $\chi_\varDelta \in \mathcal{D}(C_\phi)$ for any $\varDelta \in \mathscr{A}$ such that $\varDelta \subseteq Y_n$. Substituting $f = \chi_\varDelta$ into (7), we get $\int_{Y_n} |\mathsf{h}_\phi - c^2| \, \mathrm{d}\mu < \infty$ and $\int_\varDelta (\mathsf{h}_\phi - c^2) \, \mathrm{d}\mu \geqslant 0$ for every $\varDelta \in \mathscr{A}$ such that $\varDelta \subseteq Y_n$. This implies that $\mathsf{h}_\phi - c^2 \geqslant 0$ a.e. $[\mu]$ on Y_n. Since $Y_k \nearrow Y$ as $k \to \infty$, where $Y = \{x \in X \colon \mathsf{h}_\phi(x) < \infty\}$, we conclude that $\mathsf{h}_\phi \geqslant c^2$ a.e. $[\mu]$. The reverse implication is obvious. $\square$

Now we collect some properties of conditional expectation that are needed in this paper. Set $\phi^{-1}(\mathscr{A}) = \{\phi^{-1}(\varDelta) \colon \varDelta \in \mathscr{A}\}$. Suppose ϕ is a nonsingular transformation of X such that $\mathsf{h}_\phi < \infty$ a.e. $[\mu]$. Then the measure $\mu|_{\phi^{-1}(\mathscr{A})}$ is σ-finite (cf. [13, Proposition 3.2]), and thus by the Radon–Nikodym theorem, for every $\mathscr{A}$-measurable function

$f\colon X \to \overline{\mathbb{R}}_+$ there exists a unique (up to sets of measure μ zero) $\phi^{-1}(\mathscr{A})$-measurable function[1] $\mathsf{E}(f)\colon X \to \overline{\mathbb{R}}_+$ such that for every $\mathscr{A}$-measurable function $g\colon X \to \overline{\mathbb{R}}_+$,

$$\int_X g \circ \phi \cdot f \,\mathrm{d}\mu = \int_X g \circ \phi \cdot \mathsf{E}(f) \,\mathrm{d}\mu. \tag{8}$$

We call $\mathsf{E}(f)$ the *conditional expectation* of f with respect to $\phi^{-1}(\mathscr{A})$ (see [48] and [13] for more information). For simplicity we do not make the dependence of $\mathsf{E}(f)$ on ϕ explicit. It is well-known that

$$\text{if } 0 \leqslant f_n \nearrow f \text{ and } f_n,\ f \text{ are } \mathscr{A}\text{-measurable, then } \mathsf{E}(f_n) \nearrow \mathsf{E}(f), \tag{9}$$

where $g_n \nearrow g$ means that for μ-a.e. $x \in X$, the sequence $\{g_n(x)\}_{n=1}^\infty$ is monotonically increasing and convergent to $g(x)$. Note that for every $\mathscr{A}$-measurable function $u\colon X \to \overline{\mathbb{R}}_+$ there exists a unique (up to sets of measure μ zero) $\mathscr{A}$-measurable function $g\colon X \to \overline{\mathbb{R}}_+$ such that $u \circ \phi = g \circ \phi$ a.e. $[\mu]$ and $g = 0$ a.e. $[\mu]$ on $X \setminus \Omega_\phi$, where $\Omega_\phi := \{\mathsf{h}_\phi > 0\}$. Indeed, by the measure transport theorem, we have $\int_{\phi^{-1}(\Delta)} u \circ \phi \,\mathrm{d}\mu = \int_\Delta u\,\mathsf{h}_\phi \,\mathrm{d}\mu = \int_{\phi^{-1}(\Delta)} (u\,\chi_{\Omega_\phi}) \circ \phi \,\mathrm{d}\mu$ for all $\Delta \in \mathscr{A}$, and thus $g = u\,\chi_{\Omega_\phi}$ has the required properties (because $\mu|_{\phi^{-1}(\mathscr{A})}$ is σ-finite). A similar argument yields the uniqueness of g. As a consequence, if $f\colon X \to \overline{\mathbb{R}}_+$ is an $\mathscr{A}$-measurable function, then $\mathsf{E}(f) = g \circ \phi$ a.e. $[\mu]$ with some $\mathscr{A}$-measurable function $g\colon X \to \overline{\mathbb{R}}_+$ such that $g = 0$ a.e. $[\mu]$ on $X \setminus \Omega_\phi$. Set $\mathsf{E}(f) \circ \phi^{-1} = g$ a.e. $[\mu]$. By the above discussion (see also [18]), this definition is correct and

$$(\mathsf{E}(f) \circ \phi^{-1}) \circ \phi = \mathsf{E}(f) \quad \text{a.e. } [\mu|_{\phi^{-1}(\mathscr{A})}]. \tag{10}$$

In particular, the following holds.

$$\begin{aligned}&\text{If } \phi \text{ is a nonsingular transformation of } X \text{ such that } 0 < \mathsf{h}_\phi < \infty \text{ a.e. } [\mu]\\ &\text{and } u, g\colon X \to \overline{\mathbb{R}}_+ \text{ are } \mathscr{A}\text{-measurable functions such that } u \circ \phi = g \circ \phi\\ &\text{a.e. } [\mu],\ \text{then } u = g \text{ a.e. } [\mu].\end{aligned} \tag{11}$$

The reader should be aware of the fact that $\mathsf{E}(\chi_X) = 1$ a.e. $[\mu]$ and

$$\mathsf{E}(\chi_X) \circ \phi^{-1} = \chi_{\{\mathsf{h}_\phi > 0\}} \text{ a.e. } [\mu]. \tag{12}$$

2. A consistency technique in subnormality

2.1. The general case

Let $(X, \mathscr{A})$ and (T, Σ) be measurable spaces and $P\colon X \times \Sigma \to [0, 1]$ be an *$\mathscr{A}$-measurable family of probability measures*, i.e.,

[1] Recall the well-known fact that a function $v\colon X \to \overline{\mathbb{R}}_+$ is $\phi^{-1}(\mathscr{A})$-measurable if and only if there exists an $\mathscr{A}$-measurable function $u\colon X \to \overline{\mathbb{R}}_+$ such that $v = u \circ \phi$.

(i) the set-function $P(x, \cdot)$ is a probability measure for every $x \in X$,
(ii) the function $P(\cdot, \sigma)$ is $\mathscr{A}$-measurable for every $\sigma \in \Sigma$.

Denote by $\mathscr{A} \otimes \Sigma$ the σ-algebra generated by the family

$$\mathscr{A} \boxtimes \Sigma := \{\Delta \times \sigma \colon \Delta \in \mathscr{A},\, \sigma \in \Sigma\}.$$

Let $\mu \colon \mathscr{A} \to \overline{\mathbb{R}}_+$ be a σ-finite measure. Then (cf. [3, Theorem 2.6.2]) there exists a unique measure ρ on $\mathscr{A} \otimes \Sigma$ such that

$$\rho(\Delta \times \sigma) = \int_{\Delta} P(x, \sigma)\mu(\mathrm{d}x), \quad \Delta \in \mathscr{A}, \sigma \in \Sigma. \tag{13}$$

Such a ρ is automatically σ-finite. Moreover, for every $\mathscr{A} \otimes \Sigma$-measurable function $f \colon X \times T \to \overline{\mathbb{R}}_+$,

$$\text{the function } X \ni x \to \int_T f(x, t) P(x,\, \mathrm{d}t) \in \overline{\mathbb{R}}_+ \text{ is } \mathscr{A}\text{-measurable} \tag{14}$$

and

$$\int_{X \times T} f \,\mathrm{d}\rho = \int_X \int_T f(x, t) P(x,\, \mathrm{d}t)\mu(\mathrm{d}x). \tag{15}$$

Let ϕ be an $\mathscr{A}$-measurable transformation of X. Define the transformation Φ of $X \times T$ by

$$\Phi(x, t) = (\phi(x), t), \quad x \in X, t \in T. \tag{16}$$

Since the σ-algebra $\{E \in \mathscr{A} \otimes \Sigma \colon \Phi^{-1}(E) \in \mathscr{A} \otimes \Sigma\}$ contains $\mathscr{A} \boxtimes \Sigma$, we deduce that the transformation Φ is $\mathscr{A} \otimes \Sigma$-measurable.

The assumptions we gather below will be used in further parts of this section.

> The triplet $(X, \mathscr{A}, \mu)$ is a σ-finite measure space, ϕ is an $\mathscr{A}$-measurable transformation of X, (T, Σ) is a measurable space and $P \colon X \times \Sigma \to [0, 1]$ is an $\mathscr{A}$-measurable family of probability measures. The measure $\rho \colon \mathscr{A} \otimes \Sigma \to \overline{\mathbb{R}}_+$ and the transformation Φ of $X \times T$ are determined by (13) and (16), respectively. (17)

We begin by establishing the basic formula that links h_ϕ and h_Φ.

Lemma 5. *Suppose (17) holds. Then the following assertions are valid.*

(i) *If ϕ is nonsingular and $P(x, \cdot) \ll P(\phi(x), \cdot)$ for μ-a.e. $x \in X$, then Φ is nonsingular.*

(ii) *If Φ is nonsingular, then so is ϕ.*

(iii) *If Φ is nonsingular and $\mathsf{h}_\phi < \infty$ a.e. $[\mu]$, then $\mathsf{h}_\Phi < \infty$ a.e. $[\rho]$ and*

$$\mathsf{h}_\phi(x)\big(\mathsf{E}(P(\cdot,\sigma)) \circ \phi^{-1}\big)(x) = \int_\sigma \mathsf{h}_\Phi(x,t)P(x,\,\mathrm{d}t) \text{ for } \mu\text{-a.e. } x \in X, \quad \sigma \in \Sigma. \tag{18}$$

Proof. (i) Take $E \in \mathscr{A} \otimes \Sigma$ such that $\rho(E) = 0$. Then, by (15), we have

$$\int_T \chi_E(x,t)P(x,\,\mathrm{d}t) = 0 \quad \text{for } \mu\text{-a.e. } x \in X.$$

Hence $\chi_E(x,t) = 0$ for $P(x,\cdot)$-a.e. $t \in T$ and for μ-a.e. $x \in X$. Since ϕ is nonsingular, we see that $\chi_E(\phi(x),t) = 0$ for $P(\phi(x),\cdot)$-a.e. $t \in T$ and for μ-a.e. $x \in X$. By our assumption, this implies that $\chi_E(\phi(x),t) = 0$ for $P(x,\cdot)$-a.e. $t \in T$ and for μ-a.e. $x \in X$. This combined with (15) implies that $\rho(\Phi^{-1}(E)) = 0$.

(ii) If $\Delta \in \mathscr{A}$ is such that $\mu(\Delta) = 0$, then by (13) we have $\rho(\Delta \times T) = \mu(\Delta) = 0$ and thus $\mu(\phi^{-1}(\Delta)) = \rho(\Phi^{-1}(\Delta \times T)) = 0$.

(iii) Applying the measure transport theorem, we obtain

$$\begin{aligned}\rho(\Phi^{-1}(\Delta\times\sigma)) &= \rho(\phi^{-1}(\Delta)\times\sigma) \overset{(13)}{=} \int_{\phi^{-1}(\Delta)} P(x,\sigma)\mu(\mathrm{d}x) \\ &\overset{(8)}{=} \int_{\phi^{-1}(\Delta)} \mathsf{E}(P(\cdot,\sigma))\,\mathrm{d}\mu \overset{(10)}{=} \int_\Delta \mathsf{h}_\phi \mathsf{E}(P(\cdot,\sigma))\circ\phi^{-1}\,\mathrm{d}\mu, \quad \Delta\in\mathscr{A}, \sigma\in\Sigma.\end{aligned} \tag{19}$$

Since Φ is nonsingular, we infer from (15) that

$$\rho(\Phi^{-1}(\Delta\times\sigma)) = \int_\Delta\int_\sigma \mathsf{h}_\Phi(x,t)P(x,\,\mathrm{d}t)\mu(\mathrm{d}x), \quad \Delta\in\mathscr{A}, \sigma\in\Sigma. \tag{20}$$

Combining (19) with (20) and using the σ-finiteness of μ, we get (18).

Since $\mathsf{h}_\phi < \infty$ a.e. $[\mu]$, there exists $\{\Delta_n\}_{n=1}^\infty \subseteq \mathscr{A}$ such that $\Delta_n \nearrow X$ as $n\to\infty$, $\mu(\Delta_k) < \infty$ and $\mathsf{h}_\phi \leqslant k$ a.e. $[\mu]$ on Δ_k for every $k \in \mathbb{N}$. Then

$$\begin{aligned}\int_{\Delta_n\times T} \mathsf{h}_\Phi\,\mathrm{d}\rho &\overset{(15)}{=} \int_{\Delta_n}\int_T \mathsf{h}_\Phi(x,t)P(x,\,\mathrm{d}t)\mu(\mathrm{d}x) \\ &\overset{(18)}{=} \int_{\Delta_n} \mathsf{h}_\phi\mathsf{E}(P(\cdot,T))\circ\phi^{-1}\,\mathrm{d}\mu \overset{(12)}{=} \int_{\Delta_n}\mathsf{h}_\phi\,\mathrm{d}\mu \leqslant n\mu(\Delta_n), \quad n\in\mathbb{N},\end{aligned} \tag{21}$$

which implies that $\mathsf{h}_\Phi < \infty$ a.e. $[\rho]$ on $\Delta_n \times T$. Since $\Delta_n\times T \nearrow X\times T$ as $n\to\infty$, we conclude that $\mathsf{h}_\Phi < \infty$ a.e. $[\rho]$. This completes the proof. □

Below we introduce the conditions (CC$_\zeta$) and (CC$_\zeta^{-1}$) (cf. Lemma 6 and Theorem 7) which play a fundamental role in this paper. We begin by proving that the first moments $\int_T \zeta(t) P(\cdot, \mathrm{d}t)$ of an $\mathscr{A}$-measurable family $P\colon X \times \Sigma \to [0,1]$ of probability measures satisfying (CC$_\zeta$) cannot vanish on a set of positive measure μ. We also calculate h_Φ.

Lemma 6. *Suppose (17) holds, ϕ is nonsingular, $\mathsf{h}_\phi < \infty$ a.e. $[\mu]$ and $\zeta\colon T \to \overline{\mathbb{R}}_+$ is a Σ-measurable function such that*[2]

$$\mathsf{E}(P(\cdot,\sigma))(x) = \frac{\int_\sigma \zeta(t) P(\phi(x), \mathrm{d}t)}{\mathsf{h}_\phi(\phi(x))} \text{ for } \mu\text{-a.e. } x \in X, \quad \sigma \in \Sigma. \tag{CC$_\zeta$}$$

Then the following three assertions hold:

(i) *$P(x, \{\zeta = 0\}) = 0$ for μ-a.e. $x \in X$, and $\zeta > 0$ a.e. $[\rho]$,*
(ii) *if $\Delta \in \mathscr{A}$ is such that $\int_T \zeta(t) P(x, \mathrm{d}t) = 0$ for μ-a.e. $x \in \Delta$, then $\mu(\Delta) = 0$,*
(iii) *Φ is nonsingular and*

$$\mathsf{h}_\Phi(x,t) = \chi_{\{\mathsf{h}_\phi > 0\}}(x)\zeta(t) \text{ for } \rho\text{-a.e. } (x,t) \in X \times T. \tag{22}$$

Proof. (i) It follows from (CC$_\zeta$) that $\mathsf{E}(P(\cdot, \{\zeta = 0\})) = 0$ a.e. $[\mu]$. Hence

$$\int_{\phi^{-1}(X)} P(x, \{\zeta = 0\}) \mu(\mathrm{d}x) = 0,$$

and thus $P(x, \{\zeta = 0\}) = 0$ for μ-a.e. $x \in X$. This in turn implies that

$$\rho(\{(x,t) \in X \times T\colon \zeta(t) = 0\}) \overset{(13)}{=} \int_X P(x, \{\zeta = 0\}) \mu(\mathrm{d}x) = 0,$$

which means that $\zeta > 0$ a.e. $[\rho]$.

(ii) If $x \in X$ is such that $\int_T \zeta(t) P(x, \mathrm{d}t) = 0$, then $P(x, \{\zeta > 0\}) = 0$. This combined with (i) implies that $P(x,T) = 0$ for μ-a.e. $x \in \Delta$. Since $P(x,T) = 1$ for every $x \in X$, we get $\mu(\Delta) = 0$.

(iii) Arguing as in (19) and using Proposition 3, we get

$$\begin{aligned} \rho(\Phi^{-1}(\Delta \times \sigma)) &= \int_{\phi^{-1}(\Delta)} \mathsf{E}(P(\cdot,\sigma)) \, \mathrm{d}\mu \\ &\overset{(\mathrm{CC}_\zeta)}{=} \int_{\phi^{-1}(\Delta)} \frac{\int_\sigma \zeta(t) P(\phi(x), \mathrm{d}t)}{\mathsf{h}_\phi(\phi(x))} \mu(\mathrm{d}x), \end{aligned}$$

[2] By (5) and (14) the right-hand side of the equality in (CC$_\zeta$) is $\mathscr{A}$-measurable a.e. $[\mu]$.

$$\overset{(6)}{=} \int_{\Delta} \chi_{\{\mathsf{h}_\phi>0\}}(x) \int_{\sigma} \zeta(t)P(x, \mathrm{d}t)\mu(\mathrm{d}x)$$

$$\overset{(15)}{=} \int_{\Delta\times\sigma} \chi_{\{\mathsf{h}_\phi>0\}}(x)\zeta(t)\,\mathrm{d}\rho(x,t), \quad \Delta \in \mathscr{A}, \sigma \in \Sigma. \tag{23}$$

It is clear that $\mathscr{P} := \mathscr{A} \boxtimes \Sigma$ is a semi-algebra such that $\sigma(\mathscr{P}) = \mathscr{A} \otimes \Sigma$. Since $\mathsf{h}_\phi < \infty$ a.e. $[\mu]$, there exists a sequence $\{\Delta_n\}_{n=1}^\infty \subseteq \mathscr{A}$ such that $\Delta_n \nearrow X$ as $n \to \infty$, $\mu(\Delta_k) < \infty$ and $\mathsf{h}_\phi \leqslant k$ a.e. $[\mu]$ on Δ_k for every $k \in \mathbb{N}$. Then

$$\rho(\Phi^{-1}(\Delta_n \times T)) \overset{(13)}{=} \mu(\phi^{-1}(\Delta_n)) = \int_{\Delta_n} \mathsf{h}_\phi\,\mathrm{d}\mu \leqslant n\mu(\Delta_n) < \infty, \quad n \in \mathbb{N}. \tag{24}$$

By (23), (24) and Lemma 1, the measures $\mathscr{A} \otimes \Sigma \ni E \to \rho(\Phi^{-1}(E)) \in \overline{\mathbb{R}}_+$ and $\mathscr{A} \otimes \Sigma \ni E \to \int_E \chi_{\{\mathsf{h}_\phi>0\}}(x)\zeta(t)\,\mathrm{d}\rho(x,t) \in \overline{\mathbb{R}}_+$ coincide. Consequently, Φ is nonsingular and, by the σ-finiteness of ρ, the equality (22) holds. □

Now we identify circumstances under which the Radon–Nikodym derivative h_Φ depends only on the second variable.

Theorem 7. *Suppose* (17) *holds, $\zeta: T \to \overline{\mathbb{R}}_+$ is a Σ-measurable function, ϕ is nonsingular and $\mathsf{h}_\phi < \infty$ a.e. $[\mu]$. Then the following assertions are equivalent:*

- (i) (CC$_\zeta$) *holds and* $\mathsf{h}_\phi > 0$ *a.e.* $[\mu]$,
- (ii) (CC$_\zeta$) *holds and* $\int_T \zeta(t)P(\cdot, \mathrm{d}t) = 0$ *a.e.* $[\mu]$ *on* $\{\mathsf{h}_\phi = 0\}$,
- (iii) (CC$_\zeta$) *holds, Φ is nonsingular and C_Φ is quasinormal,*
- (iv) *the condition below holds*

$$\mathsf{h}_\phi(x)\big(\mathsf{E}(P(\cdot,\sigma)) \circ \phi^{-1}\big)(x) = \int_\sigma \zeta(t)P(x, \mathrm{d}t) \text{ for } \mu\text{-a.e. } x \in X, \quad \sigma \in \Sigma, \tag{CC$_\zeta^{-1}$}$$

- (v) *Φ is nonsingular and $\mathsf{h}_\Phi(x,t) = \zeta(t)$ for ρ-a.e. $(x,t) \in X \times T$,*
- (vi) *Φ is nonsingular, $\mathsf{h}_\phi > 0$ a.e. $[\mu]$ and*

$$\int_\sigma \mathsf{h}_\Phi(\phi(x),t)P(\phi(x), \mathrm{d}t) = \int_\sigma \zeta(t)P(\phi(x), \mathrm{d}t) \text{ for } \mu\text{-a.e. } x \in X, \quad \sigma \in \Sigma. \tag{25}$$

Moreover, each of the conditions (i) *to* (vi) *uniquely determines ζ (up to sets of measure ρ zero) and guarantees that $0 < \zeta < \infty$ a.e. $[\rho]$.*

Proof. (i)⇒(iv) Set $H_\sigma(x) = \int_\sigma \zeta(t)P(x, \mathrm{d}t)$ for $x \in X$ and $\sigma \in \Sigma$. By (14), H_σ is $\mathscr{A}$-measurable. It follows from (CC$_\zeta$) and (10) that

$$\Big[\mathsf{h}_\phi \cdot \big(\mathsf{E}(P(\cdot,\sigma)) \circ \phi^{-1}\big)\Big] \circ \phi = H_\sigma \circ \phi \text{ a.e. } [\mu], \quad \sigma \in \Sigma.$$

This and the assumption that $\mathsf{h}_\phi > 0$ a.e. $[\mu]$ imply (CC_ζ^{-1}) (see (11)).

(iv)⇒(ii) Substituting $\sigma = T$ into (CC_ζ^{-1}), we deduce that $\int_T \zeta(t)P(\cdot, \mathrm{d}t) = 0$ a.e. $[\mu]$ on $\{\mathsf{h}_\phi = 0\}$. Composing both sides of the equality in (CC_ζ^{-1}) with ϕ and using (2) and (10), we obtain (CC_ζ).

(ii)⇒(i) Apply Lemma 6(ii) with $\Delta := \{\mathsf{h}_\phi = 0\}$.

(i)⇒(v) Note that if $f, g\colon X \to \overline{\mathbb{R}}_+$ are $\mathscr{A}$-measurable functions such that $f = g$ a.e. $[\mu]$, then $f(x)\zeta(t) = g(x)\zeta(t)$ for ρ-a.e. $(x,t) \in X \times T$. Indeed, by (15), we get

$$\int_E f(x)\zeta(t)\,\mathrm{d}\rho(x,t) = \int_X f(x) \int_T \chi_E(x,t)\zeta(t)P(x, \mathrm{d}t)\mu(\mathrm{d}x) = \int_E g(x)\zeta(t)\,\mathrm{d}\rho(x,t)$$

for every $E \in \mathscr{A} \otimes \Sigma$, which together with the σ-finiteness of ρ proves our claim. This property combined with Lemma 6(iii) implies (v).

(v)⇒(iv) Employing (15) and the σ-finiteness of μ, we deduce that for every $\sigma \in \Sigma$ and for μ-a.e. $x \in X$, $\int_\sigma \mathsf{h}_\Phi(x,t)P(x, \mathrm{d}t) = \int_\sigma \zeta(t)P(x, \mathrm{d}t)$. This and Lemma 5(iii) yield (CC_ζ^{-1}).

(v)⇒(iii) It follows from Lemma 5(iii) that $\mathsf{h}_\Phi < \infty$ a.e. $[\rho]$, and thus, by (4), C_Φ is densely defined. Using (2), we see that $\mathsf{h}_\Phi = \mathsf{h}_\Phi \circ \Phi$ a.e. $[\rho]$. Hence, by [13, Proposition 8.1], C_Φ is quasinormal. Since (v) implies (i), (CC_ζ) holds.

(iii)⇒(i) By [13, Proposition 8.1 and Corollary 6.6], C_Φ is injective. Define the mapping $U\colon L^2(\mu) \to L^2(\rho)$ by $(Uf)(x,t) = f(x)$ for $(x,t) \in X \times T$. Then, in view of (15), U is a well-defined isometric embedding such that $UC_\phi = C_\Phi U$. Hence C_ϕ is injective. It follows from [13, Proposition 6.2] that $\mathsf{h}_\phi > 0$ a.e. $[\mu]$.

(v)⇒(vi) As (v) implies (i), we get $\mathsf{h}_\phi > 0$ a.e. $[\mu]$. Applying the measure transport theorem, we see that

$$\begin{aligned}
\int_{\phi^{-1}(\Delta)} \int_\sigma \mathsf{h}_\Phi(\phi(x),t)P(\phi(x), \mathrm{d}t)\mu(\mathrm{d}x) &= \int_\Delta \int_\sigma \mathsf{h}_\phi(x)\mathsf{h}_\Phi(x,t)P(x, \mathrm{d}t)\mu(\mathrm{d}x) \\
&\overset{(15)}{=} \int_{\Delta\times\sigma} \mathsf{h}_\phi(x)\mathsf{h}_\Phi(x,t)\,\mathrm{d}\rho(x,t) \overset{(\mathrm{v})}{=} \int_{\Delta\times\sigma} \mathsf{h}_\phi(x)\zeta(t)\,\mathrm{d}\rho(x,t) \\
&= \int_{\phi^{-1}(\Delta)} \int_\sigma \zeta(t)P(\phi(x), \mathrm{d}t)\mu(\mathrm{d}x), \quad \Delta \in \mathscr{A}, \sigma \in \Sigma.
\end{aligned} \tag{26}$$

This, together with the σ-finiteness of $\mu|_{\phi^{-1}(\mathscr{A})}$, yields (vi).

(vi)⇒(i) By Lemma 5(iii), the condition (18) holds. Composing both sides of the equality in (18) with ϕ and using (2) and (10), we obtain

$$\mathsf{E}(P(\cdot,\sigma))(x) = \frac{\int_\sigma \mathsf{h}_\Phi(\phi(x),t)P(\phi(x), \mathrm{d}t)}{\mathsf{h}_\phi(\phi(x))} \text{ for } \mu\text{-a.e. } x \in X, \quad \sigma \in \Sigma.$$

This, together with (25), gives (CC_ζ).

Now we justify the "moreover" part. The uniqueness of ζ follows from the equivalence of the conditions (i) to (vi) and the equality in (v). In turn, by Lemma 5(iii) and Lemma 6(i), we see that $0 < \zeta < \infty$ a.e. $[\rho]$. This completes the proof. $\square$

Let us make two comments concerning Theorem 7.

Remark 8. a) First note that instead of proving the implication (vi)⇒(i), one can prove the implication (vi)⇒(v). The latter can be justified as follows. Since $\mathsf{h}_\phi > 0$ a.e. $[\mu]$ and $\rho(\{(x,t) \in X \times T : \mathsf{h}_\phi(x) = 0\}) = \mu(\{x \in X : \mathsf{h}_\phi(x) = 0\}$ (cf. (13)), we get $\mathsf{h}_\phi > 0$ a.e. $[\rho]$. Arguing as in (26), we see that for every $E \in \mathscr{A} \boxtimes \Sigma$,

$$\int_E \mathsf{h}_\phi(x)\mathsf{h}_\Phi(x,t)\,\mathrm{d}\rho(x,t) = \int_E \mathsf{h}_\phi(x)\zeta(t)\,\mathrm{d}\rho(x,t). \tag{27}$$

It follows from (21) that $\int_{\Delta_n \times T} \mathsf{h}_\phi(x)\mathsf{h}_\Phi(x,t)\,\mathrm{d}\rho(x,t) \leqslant n^2\mu(\Delta_n) < \infty$ for every $n \in \mathbb{N}$. Hence, by Lemma 1, the equality (27) is valid for every $E \in \mathscr{A} \otimes \Sigma$. Since ρ is σ-finite, we deduce that $\mathsf{h}_\phi(x)\mathsf{h}_\Phi(x,t) = \mathsf{h}_\phi(x)\zeta(t)$ for ρ-a.e. $(x,t) \in X \times T$. This and the fact that $\mathsf{h}_\phi > 0$ a.e. $[\rho]$ imply (v).

b) Under the assumptions of Theorem 7, if Φ is nonsingular and there exists a countable family Σ_0 of subsets of T such that $\Sigma = \sigma(\Sigma_0)$ (in particular, this is the case for $T = \mathbb{R}_+$ and $\Sigma = \mathfrak{B}(\mathbb{R}_+)$), then (25) holds if and only if

$$\mathsf{h}_\Phi(\phi(x),t) = \zeta(t) \text{ for } P(\phi(x),\cdot)\text{-a.e. } t \in T \text{ and for } \mu\text{-a.e. } x \in X. \tag{28}$$

For this, note that without loss of generality we may assume that Σ_0 is a countable algebra of sets. Suppose (25) holds. It follows from (21) that $\int_T \mathsf{h}_\Phi(x,t)P(x,\mathrm{d}t) < \infty$ for μ-a.e. $x \in X$, and thus $\int_T \mathsf{h}_\Phi(\phi(x),t)P(\phi(x),\mathrm{d}t) < \infty$ for μ-a.e. $x \in X$. Hence, there exists $X_0 \in \mathscr{A}$ such that $\mu(X \setminus X_0) = 0$, the equality in (25) holds for all $\sigma \in \Sigma_0$ and $x \in X_0$, and $\int_T \mathsf{h}_\Phi(\phi(x),t)P(\phi(x),\mathrm{d}t) < \infty$ for every $x \in X_0$. Applying Lemma 1, we conclude that the equality in (25) holds for all $\sigma \in \Sigma$ and $x \in X_0$, which implies (28). The reverse implication is obvious.

Now we state the main criterion for subnormality of unbounded densely defined composition operators written in terms of the conditions (CC_ζ) and (CC_ζ^{-1}). Note that the injectivity assumption in the hypothesis (ii) of Theorem 9 is not restrictive because each subnormal composition operator being hyponormal is injective (see [13, Corollary 6.3]; see also [36, Theorem 9d] for the bounded case).

Theorem 9. *Let $(X, \mathscr{A}, \mu)$ be a σ-finite measure space and ϕ be a nonsingular transformation of X such that C_ϕ is densely defined. Suppose there exist an $\mathscr{A}$-measurable family $P: X \times \Sigma \to [0,1]$ of probability measures on a measurable space (T, Σ) and a Σ-measurable function $\zeta: T \to \overline{\mathbb{R}}_+$ satisfying one of the following two equivalent conditions:*

(i) (CC_ζ^{-1}) *holds,*
(ii) (CC_ζ) *holds and C_ϕ is injective.*

Then C_ϕ is subnormal. Moreover, under the notation of (17), $\varPhi$ is nonsingular and $C_\varPhi$ is a quasinormal extension of C_ϕ.

Proof. Since C_ϕ is densely defined, we infer from (4) that $\mathsf{h}_\phi < \infty$ a.e. $[\mu]$. It follows from [13, Proposition 6.2] and Theorem 7 that the conditions (i) and (ii) are equivalent. Thus, we may assume that (CC_ζ^{-1}) holds. By Theorem 7, $\varPhi$ is nonsingular and $C_\varPhi$ is quasinormal. Let U be as in the proof of the implication (iii)⇒(i) of Theorem 7. Then U is an isometric embedding such that $UC_\phi = C_\varPhi U$. This, combined with the fact that quasinormal operators are subnormal (cf. [62, Theorem 2]), completes the proof. □

From now on we will concentrate on the particular cases of (CC_ζ) and (CC_ζ^{-1}) in which $T = \mathbb{R}_+$, $\varSigma = \mathfrak{B}(\mathbb{R}_+)$ and $\zeta(t) = t$ for $t \in \mathbb{R}_+$, i.e.,

$$\mathsf{E}(P(\cdot,\sigma))(x) = \frac{\int_\sigma tP(\phi(x), \mathrm{d}t)}{\mathsf{h}_\phi(\phi(x))} \text{ for } \mu\text{-a.e. } x \in X, \quad \sigma \in \mathfrak{B}(\mathbb{R}_+), \tag{CC}$$

$$\mathsf{h}_\phi(x)\big(\mathsf{E}(P(\cdot,\sigma)) \circ \phi^{-1}\big)(x) = \int_\sigma tP(x, \mathrm{d}t) \text{ for } \mu\text{-a.e. } x \in X, \quad \sigma \in \mathfrak{B}(\mathbb{R}_+). \tag{CC^{-1}}$$

We refer to (CC) as the *consistency condition* (it has been inspired by [16]).

It is worth pointing out that if $P\colon X \times \varSigma \to [0,1]$ is an $\mathscr{A}$-measurable family of probability measures satisfying (CC_ζ) (respectively, (CC_ζ^{-1})), the function ζ is injective and $\zeta(\sigma) \in \mathfrak{B}(\mathbb{R}_+)$ for every $\sigma \in \varSigma$, then, by the measure transport theorem, the mapping $\widetilde{P}\colon X \times \mathfrak{B}(\mathbb{R}_+) \to [0,1]$ given by

$$\widetilde{P}(x,\sigma) = P(x, \zeta^{-1}(\sigma)), \quad x \in X, \sigma \in \mathfrak{B}(\mathbb{R}_+),$$

is an $\mathscr{A}$-measurable family of Borel probability measures which satisfies (CC) (respectively, (CC^{-1})).

Note that if $\mathsf{h}_\phi > 0$ a.e. $[\mu]$ (in particular, this is the case for hyponormal composition operators, cf. [13, Proposition 6.1 and Corollary 6.3]), then we can modify h_ϕ so that $\mathsf{h}_\phi(x) > 0$ for every $x \in X$. Then for every $\sigma \in \mathfrak{B}(\mathbb{R}_+)$, the function $\varGamma_\sigma\colon X \to \overline{\mathbb{R}}_+$ defined by

$$\varGamma_\sigma(x) = \frac{\int_\sigma tP(x, \mathrm{d}t)}{\mathsf{h}_\phi(x)}, \quad x \in X,$$

is $\mathscr{A}$-measurable (cf. (14)) and the function $\mathsf{E}(P(\cdot,\sigma))$ can be identified with $\varGamma_\sigma \circ \phi$ whenever (CC) holds. As a consequence (cf. Theorem 7), for every $\sigma \in \mathfrak{B}(\mathbb{R}_+)$, the function $\mathsf{E}(P(\cdot,\sigma)) \circ \phi^{-1}$ can be identified with $\varGamma_\sigma$ whenever (CC^{-1}) is satisfied.

Below we show that the consistency condition, which together with injectivity is sufficient for subnormality, turns out to be necessary in the case of quasinormal composition operators.

Proposition 10. *Let $(X, \mathscr{A}, \mu)$ be a σ-finite measure space and ϕ be a nonsingular transformation of X such that C_ϕ is quasinormal. Then there exists a $\phi^{-1}(\mathscr{A})$-measurable family $P: X \times \mathfrak{B}(\mathbb{R}_+) \to [0,1]$ of probability measures which satisfies (CC). Moreover, if $\widetilde{P}: X \times \mathfrak{B}(\mathbb{R}_+) \to [0,1]$ is any $\mathscr{A}$-measurable family of probability measures satisfying (CC), then $\widetilde{P}(x, \cdot) = P(x, \cdot)$ for μ-a.e. $x \in X$.*

Proof. We can assume that $0 < \mathsf{h}_\phi < \infty$ (cf. [13, Section 6] and (4)). It follows from [13, Proposition 8.1] that $\mathsf{h}_\phi = \mathsf{h}_\phi \circ \phi$ a.e. $[\mu]$. Define the $\phi^{-1}(\mathscr{A})$-measurable family $P: X \times \mathfrak{B}(\mathbb{R}_+) \to [0,1]$ of probability measures by

$$P(x, \sigma) = \chi_\sigma(\mathsf{h}_\phi(\phi(x))), \quad x \in X, \sigma \in \mathfrak{B}(\mathbb{R}_+). \tag{29}$$

Since $\mathsf{h}_\phi = \mathsf{h}_\phi \circ \phi$ a.e. $[\mu]$ and ϕ is nonsingular, we deduce that $P(\phi(x), \sigma) = \chi_\sigma(\mathsf{h}_\phi(\phi(x)))$ for μ-a.e. $x \in X$ and for all $\sigma \in \mathfrak{B}(\mathbb{R}_+)$. This yields

$$\frac{\int_\sigma t P(\phi(x), \mathrm{d}t)}{\mathsf{h}_\phi(\phi(x))} = \chi_\sigma(\mathsf{h}_\phi(\phi(x))) \text{ for } \mu\text{-a.e. } x \in X, \quad \sigma \in \mathfrak{B}(\mathbb{R}_+). \tag{30}$$

Combining (29) and (30) shows that P satisfies (CC).

The "moreover" part follows from (29) and Corollary 18. $\square$

2.2. The bounded case

We begin by proving a "moment measurability" lemma which is a variant of [42, Lemma 1.3]. The proof of the latter contains an error which comes from using an untrue statement that characteristic functions of Borel sets on the real line are of the first Baire category. The proof of Lemma 11 is extracted from that of [15, Theorem 4.5].

Lemma 11. *Let $(X, \mathscr{A})$ be a measurable space and K be a compact subset of the complex plane $\mathbb{C}$. Suppose that $\{\vartheta_x\}_{x \in X}$ is a family of Borel probability measures on K such that*

$$\text{the map } X \ni x \mapsto \int_K z^m \bar{z}^n \vartheta_x(\mathrm{d}z) \in \mathbb{C} \text{ is } \mathscr{A}\text{-measurable for all } m, n \in \mathbb{Z}_+. \tag{31}$$

Then the function $P: X \times \mathfrak{B}(K) \ni (x, \sigma) \mapsto \vartheta_x(\sigma) \in [0,1]$ is an $\mathscr{A}$-measurable family of probability measures.

Proof. Without loss of generality we may assume that K is a rectangle of the form $K = [-r, r] \times [-r, r]$, where r is a positive real number. It follows from (31) that for every $p \in \mathbb{C}[z, \bar{z}]$ the function $X \ni x \mapsto \int_K p \, \mathrm{d}\vartheta_x \in \mathbb{C}$ is $\mathscr{A}$-measurable, where $\mathbb{C}[z, \bar{z}]$ stands for the ring of all complex polynomials in variables z and $\bar{z}$. If $f\colon K \to \mathbb{C}$ is a continuous function, then by the Stone–Weierstrass theorem, there exists a sequence $\{p_n\}_{n=1}^\infty \subseteq \mathbb{C}[z, \bar{z}]$ which converges uniformly on K to f. Hence, due to the fact that each measure ϑ_x is finite, the sequence $\{\int_K p_n \, \mathrm{d}\vartheta_x\}_{n=1}^\infty$ converges to $\int_K f \, \mathrm{d}\vartheta_x$ for every $x \in X$, which implies that the function $X \ni x \mapsto \int_K f \, \mathrm{d}\vartheta_x \in \mathbb{C}$ is $\mathscr{A}$-measurable. Take an arbitrary rectangle $L = [a_1, b_1) \times [a_2, b_2)$ with $a_1, a_2, b_1, b_2 \in \mathbb{R}$. Then there exists a sequence $\{f_n\}_{n=1}^\infty$ of continuous functions $f_n\colon K \to [0, 1]$ which converges pointwise to $\chi_{L \cap K}$. We infer from the Lebesgue dominated convergence theorem that the function $X \ni x \mapsto \vartheta_x(L \cap K) \in [0, 1]$ is $\mathscr{A}$-measurable. Set

$$\mathfrak{M} = \big\{\sigma \in \mathfrak{B}(K)\colon \text{the function } X \ni x \mapsto \vartheta_x(\sigma) \in [0, 1] \text{ is } \mathscr{A}\text{-measurable}\big\}.$$

It is easily seen that $\mathfrak{M}$ is a monotone class which contains $\varnothing$ and K, and which is closed under the operation of taking finite disjoint union of sets. Hence, the algebra Σ_0 generated by the semi-algebra of all rectangles of the form $L \cap K$ with L as above, is contained in $\mathfrak{M}$. By the monotone class theorem (cf. [3, Theorem 1.3.9]), $\mathfrak{M} = \sigma(\Sigma_0) = \mathfrak{B}(K)$, which completes the proof. □

Remark 12. Lemma 11 can be easily adapted to the N-dimensional case by allowing exponents m, n in (31) to vary over the multiindex set $\mathbb{Z}_+^N$. The proof is essentially the same.

Note that a bounded subnormal operator S always has a bounded normal extension. Indeed, by [63, Theorem 1], the spectrum of a minimal normal extension N of spectral type of S is contained in the spectrum of S which is compact; hence, by the spectral theorem, N is bounded. This means that our definition of subnormality extends that for bounded operators.

Theorem 13. *Suppose $(X, \mathscr{A}, \mu)$ is a σ-finite measure space and ϕ is a nonsingular transformation of X such that $C_\phi \in \boldsymbol{B}(L^2(\mu))$. Then the following conditions are equivalent:*

- (i) *C_ϕ is subnormal,*
- (ii) *C_ϕ is injective and there exists an $\mathscr{A}$-measurable family $P\colon X \times \mathfrak{B}(\mathbb{R}_+) \to [0, 1]$ of probability measures which satisfies (CC),*
- (ii′) *there exists an $\mathscr{A}$-measurable family $P\colon X \times \mathfrak{B}(\mathbb{R}_+) \to [0, 1]$ of probability measures which satisfies (CC^{-1}),*
- (iii) *C_ϕ is injective and there exists an $\mathscr{A}$-measurable family $P\colon X \times \mathfrak{B}(\mathbb{R}_+) \to [0, 1]$ of probability measures such that (CC) holds and the closed support of $P(x, \cdot)$ is contained in $[0, \|C_\phi\|^2]$ for μ-a.e. $x \in X$,*
- (iii′) *there exists an $\mathscr{A}$-measurable family $P\colon X \times \mathfrak{B}(\mathbb{R}_+) \to [0, 1]$ of probability measures such that (CC^{-1}) holds and the closed support of $P(x, \cdot)$ is contained in $[0, \|C_\phi\|^2]$ for μ-a.e. $x \in X$.*

The conditions above remain equivalent if the expression "for μ-a.e. $x \in X$" is replaced by "for every $x \in X$". Moreover, if C_ϕ is subnormal and $P_1, P_2\colon X \times \mathfrak{B}(\mathbb{R}_+) \to [0,1]$ are $\mathscr{A}$-measurable families of probability measures satisfying (CC), then $P_1(x,\cdot) = P_2(x,\cdot)$ for μ-a.e. $x \in X$.

Proof. By Theorem 9, (ii) is equivalent to (ii′) and (iii) is equivalent to (iii′).

(i)⇒(iii) Since subnormal operators are hyponormal, we deduce that $\mathsf{h}_\phi > 0$ a.e. $[\mu]$ (cf. [36, Theorem 9d]), and thus C_ϕ is injective. Set $K = [0, \|C_\phi\|^2]$. By [41, Corollary 4] (or rather by [15, Theorem 3.4] where $\phi(X) = X$ is not assumed), there exist a set $\varDelta_0 \in \mathscr{A}$ and a family $\{\vartheta_x\colon x \in \varDelta_0\}$ of Borel probability measures on K such that $\mu(X \setminus \varDelta_0) = 0$ and for every $x \in \varDelta_0$,

$$\mathsf{h}_{\phi^n}(x) = \int_K t^n \vartheta_x(\mathrm{d}t), \quad n \in \mathbb{Z}_+. \tag{32}$$

Setting $\mathsf{h}_{\phi^n}(x) = \chi_{\{0\}}(n)$ and $\vartheta_x(\sigma) = \chi_\sigma(0)$ for $n \in \mathbb{Z}_+$, $x \in X \setminus \varDelta_0$ and $\sigma \in \mathfrak{B}(K)$, we may assume that each h_{ϕ^n} is $\mathscr{A}$-measurable and (32) holds for all $x \in X$. By Lemma 11, the function $\widetilde{P}\colon X \times \mathfrak{B}(K) \to [0,1]$ given by

$$\widetilde{P}(x,\sigma) = \vartheta_x(\sigma), \quad x \in X, \sigma \in \mathfrak{B}(K),$$

is an $\mathscr{A}$-measurable family of probability measures. Set $T = K$ and $\varSigma = \mathfrak{B}(K)$. Let ρ and $\varPhi$ be as in Section 2.1 (with $\widetilde{P}$ in place of P). To proceed further we need [42, Lemma 2.4]. Since its proof contains an error of the same type as that mentioned in the first paragraph of Section 2.2, we provide a correction. Applying the polynomial approximation procedure given in Lambert's original proof, we get

$$\rho(\varPhi^{-1}(E)) = \int_E t \,\mathrm{d}\rho(x,t) \tag{33}$$

for every set E of the form $E = \varDelta \times (J \cap K)$, where $\varDelta \in \mathscr{A}$ and $J = [a,b)$ with $a, b \in \mathbb{R}_+$. We shall prove that (33) holds for all $E \in \mathscr{A} \otimes \mathfrak{B}(K)$. For this, denote by $\mathscr{F}$ the algebra generated by the semi-algebra $\{[a,b) \cap K\colon a, b \in \mathbb{R}_+\}$. It is clear that $\mathscr{P} := \{\varDelta \times \sigma\colon \varDelta \in \mathscr{A},\ \sigma \in \mathscr{F}\}$ is a semi-algebra such that $\sigma(\mathscr{P}) = \mathscr{A} \otimes \mathfrak{B}(K)$ (because $\sigma(\mathscr{F}) = \mathfrak{B}(K)$). By [45, Proposition I-6-1], the equality (33) holds for all $E \in \mathscr{P}$. Note that $\rho(\varPhi^{-1}(\varDelta \times K)) = \int_\varDelta \mathsf{h}_\phi \,\mathrm{d}\mu < \infty$ whenever $\mu(\varDelta) < \infty$. As μ is σ-finite, an application of Lemma 1 shows that (33) holds for all $E \in \mathscr{A} \otimes \mathfrak{B}(K)$. This means that $\varPhi$ is nonsingular and $\mathsf{h}_\varPhi(x,t) = t$ for ρ-a.e. $(x,t) \in X \times K$. Applying Theorem 7 with $\zeta(t) := t$ for $t \in K$ yields

$$\mathsf{E}(\widetilde{P}(\cdot,\sigma))(x) = \frac{\int_\sigma t\widetilde{P}(\phi(x), \mathrm{d}t)}{\mathsf{h}_\phi(\phi(x))} \text{ for } \mu\text{-a.e. } x \in X, \quad \sigma \in \mathfrak{B}(K).$$

Setting $P(x, \sigma) = \widetilde{P}(x, \sigma \cap K)$ for $x \in X$ and $\sigma \in \mathfrak{B}(\mathbb{R}_+)$ shows that (iii) is satisfied. Note that the closed support of $P(x, \cdot)$ is contained in $[0, \|C_\phi\|^2]$ for every $x \in X$.

(iii)⇒(ii) Obvious.

(ii)⇒(i) Apply Theorem 9.

The "moreover" part follows from (iii) and Corollary 18. □

2.3. The consistency condition

The consistency condition is the subject of our investigations in this section. The following assumptions will be often used.

The triplet $(X, \mathscr{A}, \mu)$ is a σ-finite measure space, ϕ is a nonsingular transformation of X such that $\mathsf{h}_\phi < \infty$ a.e. $[\mu]$ and $P: X \times \mathfrak{B}(\mathbb{R}_+) \to [0, 1]$ is an $\mathscr{A}$-measurable family of probability measures. (34)

Lemma 14. *Suppose (34) holds. Then (CC) is equivalent to each of the following three conditions*:

(i) $\mathsf{E}\big(\int_0^\infty f(t) P(\cdot, \mathrm{d}t)\big)(x) = \frac{\int_0^\infty t f(t) P(\phi(x), \mathrm{d}t)}{\mathsf{h}_\phi(\phi(x))}$ *for μ-a.e. $x \in X$ and for every Borel function $f: \mathbb{R}_+ \to \overline{\mathbb{R}}_+$*,

(ii) $P(x, \{0\}) = 0$ *and* $\mathsf{E}\big(\int_\sigma \frac{1}{t} P(\cdot, \mathrm{d}t)\big)(x) = \frac{P(\phi(x), \sigma)}{\mathsf{h}_\phi(\phi(x))}$ *for μ-a.e. $x \in X$ and for every $\sigma \in \mathfrak{B}(\mathbb{R}_+)$*,

(iii) $P(x, \{0\}) = 0$ *and* $\mathsf{E}\big(\int_0^\infty \frac{g(t)}{t} P(\cdot, \mathrm{d}t)\big)(x) = \frac{\int_0^\infty g(t) P(\phi(x), \mathrm{d}t)}{\mathsf{h}_\phi(\phi(x))}$ *for μ-a.e. $x \in X$ and for every Borel function $g: \mathbb{R}_+ \to \overline{\mathbb{R}}_+$*,

where $\int_0^\infty h(t) P(\cdot, \mathrm{d}t)$ is understood as a function $X \ni x \to \int_0^\infty h(t) P(x, \mathrm{d}t) \in \overline{\mathbb{R}}_+$ whenever $h: \mathbb{R}_+ \to \overline{\mathbb{R}}_+$ is a Borel function. Moreover, if (CC) holds, then

$$\mathsf{E}\bigg(\int_0^\infty \frac{1}{t} P(\cdot, \mathrm{d}t)\bigg)(x) = \frac{1}{\mathsf{h}_\phi(\phi(x))} < \infty \text{ for } \mu\text{-a.e. } x \in X.$$

Proof. Since each Borel function $f: \mathbb{R}_+ \to \overline{\mathbb{R}}_+$ is a pointwise limit of an increasing sequence of nonnegative Borel simple functions, one can show that (CC) implies (i) by applying the Lebesgue monotone convergence theorem as well as the additivity and the monotone continuity of the conditional expectation (see (9)). The same argument can be used to prove that (ii) implies (iii). It is obvious that (iii) implies (ii) and that (i) implies (CC).

(i)⇒(iii) By Lemma 6(i), $P(x, \{0\}) = 0$ for μ-a.e. $x \in X$. Thus, if $g: \mathbb{R}_+ \to \overline{\mathbb{R}}_+$ is a Borel function, then, by applying (i) to the Borel function $f(t) = g(t)/t$, we obtain (iii).

(iii)⇒(i) Apply (iii) to $g(t) = t f(t)$.

The "moreover" part follows from (5) and (iii) applied to $g(t) \equiv 1$. □

The equality (35) below appeared in [42, Lemma 1.2] under the assumption that ϕ is surjective and C_ϕ is bounded. For self-containedness, we include its proof (essentially the same as that of Lambert's one).

Lemma 15. *If $(X, \mathscr{A}, \mu)$ is a σ-finite measure space and ϕ is a nonsingular transformation of X such that $\mathsf{h}_\phi < \infty$ a.e. $[\mu]$, then*

$$\mathsf{h}_{\phi^{n+1}} = \mathsf{h}_\phi \cdot \mathsf{E}(\mathsf{h}_{\phi^n}) \circ \phi^{-1} \text{ a.e. } [\mu] \text{ for all } n \in \mathbb{Z}_+, \tag{35}$$

$$\mathsf{h}_{\phi^{n+1}} \circ \phi = \mathsf{h}_\phi \circ \phi \cdot \mathsf{E}(\mathsf{h}_{\phi^n}) \text{ a.e. } [\mu] \text{ for all } n \in \mathbb{Z}_+. \tag{36}$$

Proof. In view of the measure transport theorem, we have

$$\begin{aligned}\mu(\phi^{-(n+1)}(\varDelta)) = \mu(\phi^{-n}(\phi^{-1}(\varDelta))) &= \int_{\phi^{-1}(\varDelta)} \mathsf{h}_{\phi^n}\,\mathrm{d}\mu \\ &\overset{(8)}{=} \int_{\phi^{-1}(\varDelta)} \mathsf{E}(\mathsf{h}_{\phi^n})\,\mathrm{d}\mu \overset{(10)}{=} \int_{\varDelta} \mathsf{h}_\phi \cdot \mathsf{E}(\mathsf{h}_{\phi^n}) \circ \phi^{-1}\,\mathrm{d}\mu, \quad \varDelta \in \mathscr{A},\end{aligned}$$

which yields (35). By (10) and (2), the condition (36) follows from (35). □

Remark 16. Using (35), we can express h_{ϕ^n} in terms of h_ϕ by iterating the multiplication, the conditional expectation and the operation $\mathsf{E}(g) \circ \phi^{-1}$. Unfortunately, the so-obtained formulas are rather complicated (for example, $\mathsf{h}_{\phi^2} = \mathsf{h}_\phi \cdot \mathsf{E}(\mathsf{h}_\phi) \circ \phi^{-1}$ a.e. $[\mu]$, $\mathsf{h}_{\phi^3} = \mathsf{h}_\phi \cdot \mathsf{E}\big(\mathsf{h}_\phi \cdot \mathsf{E}(\mathsf{h}_\phi) \circ \phi^{-1}\big) \circ \phi^{-1}$ a.e. $[\mu]$ and so on).

As shown below, under the assumption that $\mathsf{h}_\phi > 0$ a.e. $[\mu]$, an $\mathscr{A}$-measurable family P of probability measures satisfying (CC) has the property that the "moments" of $P(x, \cdot)$ coincide with $\{\mathsf{h}_{\phi^n}(x)\}_{n=0}^\infty$ for μ-a.e. $x \in X$. This fact plays an essential role in the present paper as well as in the proof of the new characterization of quasinormal composition operators given in [14].

Theorem 17. *Suppose (34) and (CC) hold, and $\mathsf{h}_\phi > 0$ a.e. $[\mu]$. Then*

$$\mathsf{h}_{\phi^n}(x) = \int_0^\infty t^n P(x, \mathrm{d}t) \text{ for } \mu\text{-a.e. } x \in X, \quad n \in \mathbb{Z}_+. \tag{37}$$

Moreover, if C_ϕ^n is densely defined for every $n \in \mathbb{Z}_+$, then $\{\mathsf{h}_{\phi^n}(x)\}_{n=0}^\infty$ is a Stieltjes moment sequence with a representing measure $P(x, \cdot)$ for μ-a.e. $x \in X$.

Proof. To prove (37), we use an induction on n. Set $H_n(x) = \int_0^\infty t^n P(x, \mathrm{d}t)$ for $x \in X$ and $n \in \mathbb{Z}_+$. By (14), the function $H_n \colon X \to \overline{\mathbb{R}}_+$ is $\mathscr{A}$-measurable for every $n \in \mathbb{Z}_+$. Since $P(x, \cdot)$, $x \in X$, are probability measures, we deduce that $H_0(x) = 1$ for all $x \in X$, and thus $H_0 = \mathsf{h}_{\phi^0}$ a.e. $[\mu]$. Suppose that $H_n = \mathsf{h}_{\phi^n}$ a.e. $[\mu]$ for a fixed $n \in \mathbb{Z}_+$. Then, by Lemma 14(i), applied to $f(t) = t^n$, we have

$$\begin{aligned} H_{n+1}(\phi(x)) &= \int_0^\infty t^n t P(\phi(x), \mathrm{d}t) = \mathsf{h}_\phi(\phi(x)) \mathsf{E}\big(H_n\big)(x) \\ &= \mathsf{h}_\phi(\phi(x)) \mathsf{E}\big(\mathsf{h}_{\phi^n}\big)(x) \overset{(36)}{=} \mathsf{h}_{\phi^{n+1}}(\phi(x)) \quad \text{for } \mu\text{-a.e. } x \in X. \end{aligned}$$

Applying (11), we get $H_{n+1} = \mathsf{h}_{\phi^{n+1}}$ a.e. $[\mu]$, which yields (37).

The "moreover" part follows from (37) and the fact that under our density assumption, $\mathsf{h}_{\phi^n}(x) < \infty$ for μ-a.e. $x \in X$ and for every $n \in \mathbb{Z}_+$ (cf. [13, Corollary 4.5]). $\square$

Regarding Theorem 17, it is worth mentioning that C_ϕ^n is densely defined for every $n \in \mathbb{Z}_+$ if and only if $\mathcal{D}^\infty(C_\phi)$ is dense in $L^2(\mu)$ (cf. [13, Theorem 4.7]).

Corollary 18. *Suppose (34) and (CC) hold, $\mathsf{h}_\phi > 0$ a.e. $[\mu]$ and the measure $P(x, \cdot)$ is determinate for μ-a.e. $x \in X$. If $\widetilde{P}: X \times \mathfrak{B}(\mathbb{R}_+) \to [0,1]$ is any $\mathscr{A}$-measurable family of probability measures which satisfies (CC), then $\widetilde{P}(x, \cdot) = P(x, \cdot)$ for μ-a.e. $x \in X$.*

The proof of the following corollary is patterned on that of the assertion (b) of [13, Lemma 10.1].

Corollary 19. *Assume that (34) and (CC) hold, and $\mathsf{h}_\phi > 0$ a.e. $[\mu]$. Then $C_\phi^n = C_{\phi^n}$ for every $n \in \mathbb{Z}_+$.*

Proof. By (3.5) and (3.6) in [13], we have $\mathcal{D}(C_{\phi^n}) = L^2((1 + \mathsf{h}_{\phi^n})\,\mathrm{d}\mu)$ and $\mathcal{D}(C_\phi^n) = L^2((\sum_{j=0}^n \mathsf{h}_{\phi^j})\,\mathrm{d}\mu)$, and thus $C_\phi^n \subseteq C_{\phi^n}$. Since $P(x, \cdot)$, $x \in X$, are probability measures, we deduce from (37) that for μ-a.e. $x \in X$,

$$\begin{aligned} \sum_{j=0}^n \mathsf{h}_{\phi^j}(x) &= \int_0^\infty \Big(\sum_{j=0}^n t^j\Big) P(x, \mathrm{d}t) \\ &= \int_{[0,1]} \Big(\sum_{j=0}^n t^j\Big) P(x, \mathrm{d}t) + \int_{(1,\infty)} \Big(\sum_{j=0}^n t^j\Big) P(x, \mathrm{d}t) \\ &\leqslant (n+1)(1 + \mathsf{h}_{\phi^n}(x)), \end{aligned}$$

which implies that $\mathcal{D}(C_{\phi^n}) \subseteq \mathcal{D}(C_\phi^n)$. This completes the proof. $\square$

Remark 20. If C_ϕ^n is not densely defined for some integer $n \geqslant 1$, then h_{ϕ^n} takes the value ∞ on a set of positive measure (cf. (4)), which in view of (37) may lead to infinite moments. We say that a sequence $\boldsymbol{\gamma} = \{\gamma_n\}_{n=0}^\infty \subseteq \overline{\mathbb{R}}_+$ is a *pseudo-Stieltjes moment sequence* if there exists a finite Borel measure ν on $\mathbb{R}_+$, called a *representing measure* of $\boldsymbol{\gamma}$, such that $\gamma_n = \int_0^\infty s^n \nu(\mathrm{d}x)$ for all $n \in \mathbb{Z}_+$. If $\gamma_k = \infty$ for some $k \in \mathbb{N}$, then there exists a unique $k_\infty(\boldsymbol{\gamma}) \in \mathbb{N}$ such that $\gamma_k = \infty$ for every integer $k \geqslant k_\infty(\boldsymbol{\gamma})$, and $\gamma_k < \infty$ for every nonnegative integer $k < k_\infty(\boldsymbol{\gamma})$. It is easily seen that for every $k \in \mathbb{N}$, there exists a pseudo-Stieltjes moment sequence $\boldsymbol{\gamma}$ such that $k_\infty(\boldsymbol{\gamma}) = k$ (e.g., the one represented by the measure $\nu = \sum_{j=1}^\infty \frac{1}{j^{k+1}} \delta_j$). Note that if $\boldsymbol{\gamma}$ is a pseudo-Stieltjes moment sequence which is not a Stieltjes moment sequence, then it has infinitely many representing measures (i.e., $\boldsymbol{\gamma}$ is indeterminate). Indeed, let ν be a representing measure of $\boldsymbol{\gamma}$. Since the truncated Stieltjes moment problem (with the unknown Borel measure ϑ on $\mathbb{R}_+$)

$$\gamma_n = \int_0^\infty s^n \vartheta(\mathrm{d}s), \quad n = 0, \ldots, k_\infty(\boldsymbol{\gamma}) - 1, \tag{38}$$

has a solution $\vartheta = \nu$, we infer from [24, Theorem 3.6] that there exists a Borel measure τ on $\mathbb{R}_+$ with finite support such that (38) holds for $\vartheta = \tau$. Given $\alpha \in (0, 1)$, we set $\nu_\alpha = \alpha\tau + (1-\alpha)\nu$. It is clear that the measure ν_α satisfies (38) and that $\int_0^\infty s^n \,\mathrm{d}\nu_\alpha = \infty$ for all integers $n \geqslant k_\infty(\boldsymbol{\gamma})$. Hence ν_α represents $\boldsymbol{\gamma}$ and, as easily seen, the mapping $\alpha \mapsto \nu_\alpha$ is injective.

Remark 21. Theorem 17 suggests the method of looking for an $\mathscr{A}$-measurable family P of Borel probability measures on $\mathbb{R}_+$ which satisfies (CC). First, we verify whether $\{\mathsf{h}_{\phi^n}(x)\}_{n=0}^\infty$ is a pseudo-Stieltjes moment sequence for μ-a.e. $x \in X$ (cf. Remark 20). If this is the case, then we select a family $\{\vartheta_x\}_{x \in X}$ of Borel probability measures on $\mathbb{R}_+$ such that ϑ_x is a representing measure of $\{\mathsf{h}_{\phi^n}(x)\}_{n=0}^\infty$ for μ-a.e. $x \in X$, and then verify whether the family $P\colon X \times \mathfrak{B}(\mathbb{R}_+) \to [0, 1]$ of probability measures given by $P(x, \sigma) = \vartheta_x(\sigma)$ for $x \in X$ and $\sigma \in \mathfrak{B}(\mathbb{R}_+)$ is $\mathscr{A}$-measurable and satisfies (CC). This method works perfectly well in some cases (see e.g., Theorem 32 and Example 42). Unfortunately, it may break down even if $\{\mathsf{h}_{\phi^n}(x)\}_{n=0}^\infty$ is a Stieltjes moment sequence for μ-a.e. $x \in X$. Indeed, there exists a non-subnormal injective composition operator C_ϕ in $L^2(\mu)$ such that $\overline{\mathcal{D}^\infty(C_\phi)} = L^2(\mu)$ and $\{\mathsf{h}_{\phi^n}(x)\}_{n=0}^\infty$ is a Stieltjes moment sequence for μ-a.e. $x \in X$ (cf. [39, Theorem 4.3.3] and [13, Section 11]). In view of Theorem 9, for such C_ϕ there is no possibility to select P with the desired properties.

Our next aim is to show that the condition (CC) behaves well with respect to the operation of taking powers of composition operators. We begin by proving an auxiliary result on conditional expectation which is of some independent interest in itself. Given a σ-finite measure space $(X, \mathscr{A}, \mu)$, a nonsingular transformation ϕ of X and a positive integer n such that $\mathsf{h}_{\phi^n} < \infty$ a.e. $[\mu]$, we write $\mathsf{E}_n(f)$ for the conditional expectation of an $\mathscr{A}$-measure function $f\colon X \to \overline{\mathbb{R}}_+$ with respect to the sub σ-algebra $(\phi^n)^{-1}(\mathscr{A})$ of $\mathscr{A}$. In view of the discussion in the last paragraph of Preliminaries, the expression $\mathsf{E}_n(f) \circ \phi^{-n} := \mathsf{E}_n(f) \circ (\phi^n)^{-1}$ makes sense.

Lemma 22. *Let $(X, \mathscr{A}, \mu)$ be a σ-finite measure space, ϕ be a nonsingular transformation of X and n be a positive integer such that $\mathsf{h}_\phi, \mathsf{h}_{\phi^n}, \mathsf{h}_{\phi^{n+1}} < \infty$ a.e. $[\mu]$. Then, for every $\mathscr{A}$-measurable function $f \colon X \to \overline{\mathbb{R}}_+$, the following hold:*

(i) $\mathsf{h}_{\phi^{n+1}} \cdot \mathsf{E}_{n+1}(f) \circ \phi^{-(n+1)} = \mathsf{h}_{\phi^n} \cdot \mathsf{E}_n(\mathsf{h}_\phi \cdot \mathsf{E}(f) \circ \phi^{-1}) \circ \phi^{-n}$ *a.e.* $[\mu]$,

(ii) $\mathsf{h}_{\phi^{n+1}} \cdot \mathsf{E}_{n+1}(f) \circ \phi^{-(n+1)} = \mathsf{h}_\phi \cdot \mathsf{E}(\mathsf{h}_{\phi^n} \cdot \mathsf{E}_n(f) \circ \phi^{-n}) \circ \phi^{-1}$ *a.e.* $[\mu]$.

Proof. (i) Note that

$$\begin{aligned}
\int\limits_{\phi^{-(n+1)}(\varDelta)} f \, \mathrm{d}\mu &= \int\limits_X \chi_{\phi^{-n}(\varDelta)} \circ \phi \cdot \mathsf{E}(f) \, \mathrm{d}\mu \\
&\overset{(10)}{=} \int\limits_{\phi^{-n}(\varDelta)} \mathsf{h}_\phi \cdot \mathsf{E}(f) \circ \phi^{-1} \, \mathrm{d}\mu \\
&= \int\limits_X \chi_\varDelta \circ \phi^n \cdot \mathsf{E}_n(\mathsf{h}_\phi \cdot \mathsf{E}(f) \circ \phi^{-1}) \, \mathrm{d}\mu \\
&\overset{(10)}{=} \int\limits_\varDelta \mathsf{h}_{\phi^n} \cdot \mathsf{E}_n(\mathsf{h}_\phi \cdot \mathsf{E}(f) \circ \phi^{-1}) \circ \phi^{-n} \, \mathrm{d}\mu, \quad \varDelta \in \mathscr{A},
\end{aligned} \tag{39}$$

and

$$\begin{aligned}
\int\limits_{\phi^{-(n+1)}(\varDelta)} f \, \mathrm{d}\mu &= \int\limits_X \chi_\varDelta \circ \phi^{n+1} \cdot \mathsf{E}_{n+1}(f) \, \mathrm{d}\mu \\
&\overset{(10)}{=} \int\limits_\varDelta \mathsf{h}_{\phi^{n+1}} \cdot \mathsf{E}_{n+1}(f) \circ \phi^{-(n+1)} \, \mathrm{d}\mu, \quad \varDelta \in \mathscr{A}.
\end{aligned} \tag{40}$$

Hence (i) follows from (39), (40) and the σ-finiteness of μ.

(ii) Similarly, the equalities

$$\begin{aligned}
\int\limits_{\phi^{-(n+1)}(\varDelta)} f \, \mathrm{d}\mu &= \int\limits_X \chi_{\phi^{-1}(\varDelta)} \circ \phi^n \cdot \mathsf{E}_n(f) \, \mathrm{d}\mu \\
&\overset{(10)}{=} \int\limits_X \chi_{\phi^{-1}(\varDelta)} \cdot \mathsf{h}_{\phi^n} \cdot \mathsf{E}_n(f) \circ \phi^{-n} \, \mathrm{d}\mu \\
&= \int\limits_X \chi_\varDelta \circ \phi \cdot \mathsf{E}(\mathsf{h}_{\phi^n} \cdot \mathsf{E}_n(f) \circ \phi^{-n}) \, \mathrm{d}\mu \\
&\overset{(10)}{=} \int\limits_\varDelta \mathsf{h}_\phi \cdot \mathsf{E}(\mathsf{h}_{\phi^n} \cdot \mathsf{E}_n(f) \circ \phi^{-n}) \circ \phi^{-1} \, \mathrm{d}\mu, \quad \varDelta \in \mathscr{A},
\end{aligned}$$

combined with (40), give (ii). This completes the proof. □

If $f \equiv 1$, then, in view of (12), the formulas (i) and (ii) of Lemma 22 take the following forms (see (35) where h_{ϕ^n} and $\mathsf{h}_{\phi^{n+1}}$ are not assumed to be finite a.e. $[\mu]$)

$$\mathsf{h}_{\phi^{n+1}} = \mathsf{h}_{\phi^n} \cdot \mathsf{E}_n(\mathsf{h}_\phi) \circ \phi^{-n} = \mathsf{h}_\phi \cdot \mathsf{E}(\mathsf{h}_{\phi^n}) \circ \phi^{-1} \text{ a.e. } [\mu]. \tag{41}$$

Under more restrictive assumptions on ϕ, equalities (41) appeared in [29, p. 166].

Proposition 23. *Let $(X, \mathscr{A}, \mu)$ be a σ-finite measure space, ϕ be a nonsingular transformation of X such that $0 < \mathsf{h}_\phi < \infty$ a.e. $[\mu]$. Suppose $P: X \times \mathfrak{B}(\mathbb{R}_+) \to [0,1]$ is an $\mathscr{A}$-measurable family of probability measures which satisfies (CC). Let $n \in \mathbb{N}$ be such that $\mathsf{h}_{\phi^n} < \infty$ a.e. $[\mu]$. Then for every $j = 1, \ldots, n$, $0 < \mathsf{h}_{\phi^j} < \infty$ a.e. $[\mu]$ and (CC) holds with (ϕ^j, E_j, P_j) in place of (ϕ, E, P), where $P_j: X \times \mathfrak{B}(\mathbb{R}_+) \to [0,1]$ is an $\mathscr{A}$-measurable family of probability measures defined by*

$$P_j(x, \sigma) = P(x, \eta_j^{-1}(\sigma)), \quad x \in X, \sigma \in \mathfrak{B}(\mathbb{R}_+),$$

with $\eta_j: \mathbb{R}_+ \ni t \mapsto t^j \in \mathbb{R}_+$.

Proof. It follows from Corollary 19 that $C_\phi^j = C_{\phi^j}$ for $j = 1, \ldots, n$. This together with [13, Section 6] and (4) implies that $0 < \mathsf{h}_{\phi^j} < \infty$ a.e. $[\mu]$ for $j = 1, \ldots, n$. Note that if $j \in \{1, \ldots, n\}$, then (CC) holds with (ϕ^j, E_j, P_j) in place of (ϕ, E, P) if and only if

$$\mathsf{E}_j(P(\cdot, \sigma))(x) = \frac{\int_\sigma t^j P(\phi^j(x), \mathrm{d}t)}{\mathsf{h}_{\phi^j}(\phi^j(x))} \text{ for } \mu\text{-a.e. } x \in X, \quad \sigma \in \mathfrak{B}(\mathbb{R}_+). \tag{42}$$

We use induction to prove that (42) holds for every $j \in \{1, \ldots, n\}$. The case of $j = 1$ is obvious. Assume that $n \geqslant 2$ and (42) holds for a fixed $j \in \{1, \ldots, n-1\}$. Then, by (10) and (11) applied to ϕ^j in place of ϕ, we deduce from (42) that

$$\mathsf{h}_{\phi^j}(x) \cdot (\mathsf{E}_j(P(\cdot, \sigma)) \circ \phi^{-j})(x) = \int_\sigma t^j P(x, \mathrm{d}t) \text{ for } \mu\text{-a.e. } x \in X. \tag{43}$$

Applying Lemma 22(ii) with j in place of n and using (2) and (10), we see that

$$\begin{aligned} &\mathsf{h}_{\phi^{j+1}}(\phi^{j+1}(x)) \cdot \mathsf{E}_{j+1}(P(\cdot, \sigma))(x) \\ &\quad = \mathsf{h}_\phi(\phi^{j+1}(x)) \cdot \big(\mathsf{E}\big(\mathsf{h}_{\phi^j} \cdot \mathsf{E}_j(P(\cdot, \sigma)) \circ \phi^{-j}\big) \circ \phi^j\big)(x) \\ &\quad \overset{(43)}{=} \mathsf{h}_\phi(\phi^{j+1}(x)) \cdot \Big(\mathsf{E}\Big(\int_\sigma t^j P(\cdot, \mathrm{d}t)\Big) \circ \phi^j\Big)(x) \\ &\quad \overset{(\dagger)}{=} \mathsf{h}_\phi(\phi^{j+1}(x)) \cdot \frac{\int_\sigma t^{j+1} P(\phi^{j+1}(x), \mathrm{d}t)}{\mathsf{h}_\phi(\phi^{j+1}(x))} \\ &\quad = \int_\sigma t^{j+1} P(\phi^{j+1}(x), \mathrm{d}t) \text{ for } \mu\text{-a.e. } x \in X, \end{aligned}$$

where the equality (†) follows from Lemma 14(i) and (2). Hence, (42) holds for $j+1$ in place of j. This completes the proof. □

2.4. The strong consistency condition

Under the assumptions of (34), we say that P satisfies the *strong consistency condition* if

$$P(x,\sigma) = \frac{\int_\sigma t P(\phi(x), \mathrm{d}t)}{\mathsf{h}_\phi(\phi(x))} \text{ for } \mu\text{-a.e. } x \in X, \quad \sigma \in \mathfrak{B}(\mathbb{R}_+). \tag{SCC}$$

Some characterizations of (SCC) can be easily obtained by adapting Lemma 14 and its proof to the present context. It is clear that P satisfies (SCC) if and only if it satisfies (CC) and the following equality

$$\mathsf{E}(P(\cdot,\sigma))(x) = P(x,\sigma) \text{ for } \mu\text{-a.e. } x \in X, \quad \sigma \in \mathfrak{B}(\mathbb{R}_+). \tag{44}$$

Of course, (44) is valid if $\phi^{-1}(\mathscr{A}) = \mathscr{A}$. The latter holds if ϕ is injective and $\mathscr{A}$-bimeasurable (i.e., ϕ is $\mathscr{A}$-measurable and $\phi(\Delta) \in \mathscr{A}$ for every $\Delta \in \mathscr{A}$). In particular, this is the case for matrix symbols (cf. Section 3.1). Note also that each quasinormal composition operator satisfies (SCC) with some P (cf. Proposition 10).

Now we show that if a measurable family $P: X \times \mathfrak{B}(\mathbb{R}_+) \to [0,1]$ satisfies (SCC), then all negative moments of the measure $P(x,\cdot)$ are finite for μ-a.e. $x \in X$.

Proposition 24. *Suppose (34) and (SCC) hold. Then $P(x,\{0\}) = 0$ for μ-a.e. $x \in X$ and the following equalities are valid for every Borel function $f: \mathbb{R}_+ \to \overline{\mathbb{R}}_+$:*

$$\int_0^\infty f(t) P(x, \mathrm{d}t) = \frac{\int_0^\infty f(t) t^n P(\phi^n(x), \mathrm{d}t)}{\prod_{j=1}^n \mathsf{h}_\phi(\phi^j(x))} \text{ for } \mu\text{-a.e. } x \in X, \quad n \in \mathbb{N}. \tag{45}$$

In particular, the following equalities hold for all $n \in \mathbb{N}$ and $m \in \mathbb{Z}$:

(i) $\int_\sigma t^m P(x, \mathrm{d}t) = \frac{\int_\sigma t^{m+n} P(\phi^n(x), \mathrm{d}t)}{\prod_{j=1}^n \mathsf{h}_\phi(\phi^j(x))}$ *for μ-a.e. $x \in X$ and every $\sigma \in \mathfrak{B}(\mathbb{R}_+)$,*
(ii) $\int_\sigma t^m P(x, \mathrm{d}t) = \frac{\int_\sigma t^{m+n} P(\phi^n(x), \mathrm{d}t)}{\int_0^\infty t^n P(\phi^n(x), \mathrm{d}t)}$ *for μ-a.e. $x \in X$ and every $\sigma \in \mathfrak{B}(\mathbb{R}_+)$,*
(iii) $\int_\sigma \frac{1}{t^n} P(x, \mathrm{d}t) = \frac{P(\phi^n(x),\sigma)}{\prod_{j=1}^n \mathsf{h}_\phi(\phi^j(x))}$ *for μ-a.e. $x \in X$ and every $\sigma \in \mathfrak{B}(\mathbb{R}_+)$,*
(iv) $\int_0^\infty t^n P(\phi^n(x), \mathrm{d}t) = \prod_{j=1}^n \mathsf{h}_\phi(\phi^j(x))$ *for μ-a.e. $x \in X$,*
(v) $\int_0^\infty \frac{1}{t^n} P(x, \mathrm{d}t) = \frac{1}{\prod_{j=1}^n \mathsf{h}_\phi(\phi^j(x))}$ *for μ-a.e. $x \in X$.*

Moreover, if $\mathsf{h}_\phi > 0$ a.e. $[\mu]$, then $E(\mathsf{h}_{\phi^n}) = \mathsf{h}_{\phi^n}$ a.e. $[\mu]$ for every $n \in \mathbb{Z}_+$.

Proof. That $P(x,\{0\}) = 0$ for μ-a.e. $x \in X$ follows directly from (SCC). Using repeatedly (SCC) with appropriate substitutions (cf. (2)), we get

$$P(x,\sigma) = \frac{\int_\sigma tP(\phi(x), \mathrm{d}t)}{\mathsf{h}_\phi(\phi(x))} = \frac{\int_\sigma t^2 P(\phi^2(x), \mathrm{d}t)}{\mathsf{h}_\phi(\phi(x))\mathsf{h}_\phi(\phi^2(x))} = \ldots = \frac{\int_\sigma t^n P(\phi^n(x), \mathrm{d}t)}{\prod_{j=1}^n \mathsf{h}_\phi(\phi^j(x))}$$

for μ-a.e. $x \in X$ whenever $n \in \mathbb{N}$ and $\sigma \in \mathfrak{B}(\mathbb{R}_+)$. Hence, by applying [50, Theorem 1.29], we get (45).

Substituting $f(t) = t^m \chi_\sigma(t)$ into (45), we get (i). Applying (i) to $m = -n$ we obtain (iii). In turn, applying (i) to $m = 0$ and $\sigma = \mathbb{R}_+$, we get (iv). Combining (i) and (iv) gives (ii). Finally, (v) follows from (iii), applied to $\sigma = \mathbb{R}_+$.

To show the “moreover” part, assume that $\mathsf{h}_\phi > 0$ a.e. $[\mu]$. Arguing as in the proof of Lemma 14, we infer from (44) that for every Borel function $f\colon \mathbb{R}_+ \to \overline{\mathbb{R}}_+$,

$$\mathsf{E}\Big(\int_0^\infty f(t)P(\cdot, \mathrm{d}t)\Big)(x) = \int_0^\infty f(t)P(x, \mathrm{d}t) \text{ for } \mu\text{-a.e. } x \in X.$$

Substituting $f(t) = t^n$ and using Theorem 17 we complete the proof. □

Corollary 25. *If (34) and (SCC) hold, then for every $n \in \mathbb{N}$,*

$$\prod_{j=n+1}^{2n} \mathsf{h}_\phi(\phi^j(x)) \leqslant \prod_{j=1}^{n} \mathsf{h}_\phi(\phi^j(x)) \leqslant \int_0^\infty t^n P(x, \mathrm{d}t) \textit{ for } \mu\textit{-a.e. } x \in X. \tag{46}$$

Proof. By Proposition 24(v) and the Cauchy–Schwarz inequality, we have

$$\begin{aligned}\prod_{j=1}^{n} \mathsf{h}_\phi(\phi^j(x)) \int_0^\infty \frac{1}{t^n} P(x, \mathrm{d}t) = 1 &= \Big(\int_0^\infty \sqrt{t^n}\, \frac{1}{\sqrt{t^n}} P(x, \mathrm{d}t)\Big)^2 \\ &\leqslant \int_0^\infty t^n P(x, \mathrm{d}t) \int_0^\infty \frac{1}{t^n} P(x, \mathrm{d}t) \text{ for } \mu\text{-a.e. } x \in X, \quad n \in \mathbb{N}.\end{aligned}$$

Hence, the right-hand inequality in (46) holds. This, together with Proposition 24(iv), implies the left-hand inequality in (46). □

In Proposition 26 we characterize the circumstances under which the equalities $E(\mathsf{h}_{\phi^n}) = \mathsf{h}_{\phi^n}$ a.e. $[\mu]$, $n \in \mathbb{Z}_+$, hold. It is worth mentioning that the condition (iv) below resembles the formula (6.4) in [16, Lemma 6.2] which was proved for C_0-semigroups of bounded composition operators with bimeasurable symbols.

Proposition 26. *Let $(X, \mathscr{A}, \mu)$ be a σ-finite measure space and ϕ be a nonsingular transformation of X such that $\mathsf{h}_\phi < \infty$ a.e. $[\mu]$. Then the following two conditions are equivalent:*

(i) $\mathsf{E}(\mathsf{h}_{\phi^n}) = \mathsf{h}_{\phi^n}$ *a.e. $[\mu]$ for all $n \in \mathbb{N}$,*
(ii) $\mathsf{h}_{\phi^{n+1}} \circ \phi = \mathsf{h}_\phi \circ \phi \cdot \mathsf{h}_{\phi^n}$ *a.e. $[\mu]$ for all $n \in \mathbb{N}$.*

Moreover, if (i) *holds, then the following equalities are valid*:

(iii) $\mathsf{h}_{\phi^{m+n}} \circ \phi^n = \mathsf{h}_\phi \circ \phi \cdots \mathsf{h}_\phi \circ \phi^n \cdot \mathsf{h}_{\phi^m}$ *a.e.* $[\mu]$ *for all* $m \in \mathbb{Z}_+$ *and* $n \in \mathbb{N}$,
(iv) $\mathsf{h}_{\phi^{m+n}} \circ \phi^n = \mathsf{h}_{\phi^n} \circ \phi^n \cdot \mathsf{h}_{\phi^m}$ *a.e.* $[\mu]$ *for all* $m \in \mathbb{Z}_+$ *and* $n \in \mathbb{N}$,
(v) $\mathsf{h}_{\phi^n} \circ \phi^n = \mathsf{h}_\phi \circ \phi \cdots \mathsf{h}_\phi \circ \phi^n$ *a.e.* $[\mu]$ *for all* $n \in \mathbb{N}$,
(vi) $\mathsf{h}_{\phi^{n+1}} \circ \phi^n = \mathsf{h}_\phi \circ \phi^0 \cdots \mathsf{h}_\phi \circ \phi^n$ *a.e.* $[\mu]$ *for all* $n \in \mathbb{Z}_+$.

Proof. (i)⇒(ii) This is a direct consequence of (36).

(ii)⇒(i) Applying the operator of conditional expectation to both sides of the equality in (ii) and using (8), we get $\mathsf{h}_{\phi^{n+1}} \circ \phi = \mathsf{h}_\phi \circ \phi \cdot \mathsf{E}(\mathsf{h}_{\phi^n})$ a.e. $[\mu]$ for all $n \in \mathbb{N}$. This together with (ii) implies that $\mathsf{h}_\phi \circ \phi \cdot \mathsf{h}_{\phi^n} = \mathsf{h}_\phi \circ \phi \cdot \mathsf{E}(\mathsf{h}_{\phi^n})$ a.e. $[\mu]$ for all $n \in \mathbb{N}$. Since $\mathsf{h}_\phi \circ \phi > 0$ a.e. $[\mu]$, we get (i).

Now assume that (i) is satisfied. By (ii), the equality in (iii) is valid for $n = 1$ and for all $m \in \mathbb{Z}_+$. Suppose that this equality holds for a fixed $n \in \mathbb{N}$ and for all $m \in \mathbb{Z}_+$. Since the equality in (ii) is valid for $n = 0$, we see that for every $m \in \mathbb{Z}_+$,

$$\begin{aligned} \mathsf{h}_{\phi^{m+(n+1)}} \circ \phi^{n+1} &= \mathsf{h}_{\phi^{(m+1)+n}} \circ \phi^n \circ \phi = \mathsf{h}_\phi \circ \phi^2 \cdots \mathsf{h}_\phi \circ \phi^{n+1} \cdot \mathsf{h}_{\phi^{m+1}} \circ \phi \\ &\overset{\text{(ii)}}{=} \mathsf{h}_\phi \circ \phi^2 \cdots \mathsf{h}_\phi \circ \phi^{n+1} \cdot \mathsf{h}_\phi \circ \phi \cdot \mathsf{h}_{\phi^m} = \mathsf{h}_\phi \circ \phi \cdots \mathsf{h}_\phi \circ \phi^{n+1} \cdot \mathsf{h}_{\phi^m} \text{ a.e. } [\mu]. \end{aligned}$$

By induction, this implies (iii).

Substituting $m = 0$ and $m = 1$ into (iii) we get (v) and (vi), respectively. Combining (iii) with (v) gives (iv). This completes the proof. □

The following is a direct consequence of Propositions 24 and 26.

Corollary 27. *If* (34) *and* (SCC) *are satisfied and* $\mathsf{h}_\phi > 0$ *a.e.* $[\mu]$, *then* $\mathsf{h}_{\phi^{n+1}} \circ \phi = \mathsf{h}_\phi \circ \phi \cdot \mathsf{h}_{\phi^n}$ *a.e.* $[\mu]$ *for all* $n \in \mathbb{N}$.

Remark 28. Under the assumptions of Proposition 26, if additionally ϕ is a bijection whose inverse ϕ^{-1} is nonsingular (see [16, Lemma 3.1(ii)] for the possibility of weakening this assumption), then $\phi^{-1}(\mathscr{A}) = \mathscr{A}$ and thus, by Proposition 26(v),

$$\mathsf{h}_{\phi^n} = \mathsf{h}_\phi \circ \phi^0 \cdots \mathsf{h}_\phi \circ \phi^{-(n-1)} \text{ a.e. } [\mu], \quad n \in \mathbb{N}.$$

This happens for composition operators with matrix symbols (cf. Section 3.1).

The next observation is inspired by [16, Remark 6.4].

Remark 29. Note that if (34) holds, the measure $tP(x, \mathrm{d}t)$ is determinate for μ-a.e. $x \in X$ and $H_{n+1} \circ \phi = \mathsf{h}_\phi \circ \phi \cdot H_n$ a.e. $[\mu]$ for every $n \in \mathbb{Z}_+$, where $H_n(x) = \int_0^\infty t^n P(x, \mathrm{d}t)$, then (SCC) is valid. Moreover, if $\mathsf{h}_\phi > 0$ a.e. $[\mu]$, then $H_n = \mathsf{h}_{\phi^n}$ a.e. $[\mu]$ for every $n \in \mathbb{Z}_+$. Indeed, take a set $X_0 \in \mathscr{A}$ of μ-full measure such that for every $x \in X_0$, the measure

$tP(x, \mathrm{d}t)$ is determinate and $H_{n+1}(\phi(x)) = \mathsf{h}_\phi(\phi(x))H_n(x)$ for every $n \in \mathbb{Z}_+$. Then the measures $tP(\phi(x), \mathrm{d}t)$ and $P(x, \mathrm{d}t)$ are determinate for every $x \in X_0 \cap \phi^{-1}(X_0)$ (cf. [39, Lemma 2.1.1]). Since, by our assumption, the nth moments of the measures $tP(\phi(x), \mathrm{d}t)$ and $\mathsf{h}_\phi(\phi(x))P(x, \mathrm{d}t)$ coincide for all $n \in \mathbb{Z}_+$ and $x \in X_0$, and $\mu(X\setminus(X_0\cap\phi^{-1}(X_0))) = 0$, we see that (SCC) is satisfied. The "moreover" part follows from Theorem 17.

We conclude this section by showing that for bounded subnormal composition operators condition (i) of Proposition 26 holds if and only if the representing measures of $\{\mathsf{h}_{\phi^n}(x)\}_{n=0}^\infty$, $x \in X$, form a measurable family which satisfies (SCC).

Proposition 30. *Suppose (34) holds, $C_\phi \in \boldsymbol{B}(L^2(\mu))$ and*

$$\begin{aligned}&\text{for } \mu\text{-a.e. } x \in X,\ \{\mathsf{h}_{\phi^n}(x)\}_{n=0}^\infty \text{ is a Stieltjes moment sequence}\\ &\text{with a representing measure } P(x,\cdot).\end{aligned} \tag{47}$$

Then P satisfies (CC) and the following three conditions are equivalent:

(i) *P satisfies (SCC),*
(ii) *$\mathsf{E}(P(\cdot,\sigma))(x) = P(x,\sigma)$ for μ-a.e. $x \in X$ and for every $\sigma \in \mathfrak{B}(\mathbb{R}_+)$,*
(iii) *$\mathsf{E}(\mathsf{h}_{\phi^n}) = \mathsf{h}_{\phi^n}$ a.e. $[\mu]$ for every $n \in \mathbb{Z}_+$.*

Proof. First we show that P satisfies (CC). In view of (47) and Lambert's criterion (see [41]; see also [15, Theorem 3.4]), C_ϕ is subnormal. By [36, Theorem 9d] and Theorem 13, $\mathsf{h}_\phi > 0$ a.e. $[\mu]$ and there exists an $\mathscr{A}$-measurable family $\widetilde{P}: X \times \mathfrak{B}(\mathbb{R}_+) \to [0, 1]$ of probability measures which satisfies (CC) (with $\widetilde{P}$ in place of P), and which has the property that the closed support of $\widetilde{P}(x,\cdot)$ is contained in $[0, \|C_\phi\|^2]$ for μ-a.e. $x \in X$. It follows from (47) and Theorem 17 that the nth moments of the measures $P(x,\cdot)$ and $\widetilde{P}(x,\cdot)$ coincide for every $n \in \mathbb{Z}_+$ and for μ-a.e. $x \in X$. Since any Borel measure on $\mathbb{R}_+$ with compact support is determinate, we conclude that $\widetilde{P}(x,\cdot) = P(x,\cdot)$ for μ-a.e. $x \in X$. Hence P satisfies (CC).

(i)⇔(ii) This is clear, because P satisfies (CC).

(ii)⇒(iii) Apply the "moreover" part of Proposition 24.

(iii)⇒(ii) We partially follow the proof of [15, Theorem 3.4]. Without loss of generality we may assume that $\mathsf{h}_{\phi^0} = 1$, h_{ϕ^n} is $\phi^{-1}(\mathscr{A})$-measurable and $0 \leqslant \mathsf{h}_{\phi^n} < \infty$ for all $n \in \mathbb{Z}_+$. Set $Y = \bigcap_{n=0}^\infty \big\{x \in X: \mathsf{h}_{\phi^{2(n+1)}}(x) \leqslant \|C_\phi\|^4\mathsf{h}_{\phi^{2n}}(x)\big\}$. It is clear that $Y \in \phi^{-1}(\mathscr{A})$. Since for every $f \in L^2(\mu)$ and for all $n \in \mathbb{Z}_+$,

$$\int_X |f|^2\mathsf{h}_{\phi^{2(n+1)}}\,\mathrm{d}\mu = \|C_\phi^2C_\phi^{2n}f\|^2 \leqslant \|C_\phi\|^4\|C_\phi^{2n}f\|^2 = \|C_\phi\|^4\int_X |f|^2\mathsf{h}_{\phi^{2n}}\,\mathrm{d}\mu,$$

we deduce that $\mu(X \setminus Y) = 0$. Given a nonempty subset W of $\mathbb{C}$, we define the subsets Z_W and $\widetilde{Z}_W$ of X by

$$Z_W = \bigcap_{n\in\mathbb{Z}_+} \bigcap_{\lambda_0,\ldots,\lambda_n\in W} \Big\{x \in X\colon \sum_{i,j=0}^{n} \mathsf{h}_{\phi^{i+j}}(x)\lambda_i\bar{\lambda}_j \geqslant 0\Big\},$$

$$\widetilde{Z}_W = \bigcap_{n\in\mathbb{Z}_+} \bigcap_{\lambda_0,\ldots,\lambda_n\in W} \Big\{x \in X\colon \sum_{i,j=0}^{n} \mathsf{h}_{\phi^{i+j+1}}(x)\lambda_i\bar{\lambda}_j \geqslant 0\Big\}.$$

Let S be a countable and dense subset of $\mathbb{C}$. Noting that $Z_{\mathbb{C}} = Z_S$ and $\widetilde{Z}_{\mathbb{C}} = \widetilde{Z}_S$, we deduce that $Z_{\mathbb{C}}, \widetilde{Z}_{\mathbb{C}} \in \phi^{-1}(\mathscr{A})$. It follows from (47) and [4, Theorem 6.2.5] that $\mu(X \setminus Z_{\mathbb{C}}) = \mu(X \setminus \widetilde{Z}_{\mathbb{C}}) = 0$. Set $\varOmega = Y \cap Z_{\mathbb{C}} \cap \widetilde{Z}_{\mathbb{C}}$. Then $\varOmega \in \phi^{-1}(\mathscr{A})$ and $\mu(X \setminus \varOmega) = 0$. Applying [4, Theorem 6.2.5] and [67, Theorem 2], we see that for every $x \in \varOmega$ there exists a Borel probability measure ϑ_x on $K := [0, \|C_\phi\|^2]$ such that $\int_K t^n \vartheta_x(\mathrm{d}t) = \mathsf{h}_{\phi^n}(x)$ for all $n \in \mathbb{Z}_+$. It follows from Lemma 11 that the function $\varOmega \ni x \mapsto \vartheta_x(\sigma) \in [0, 1]$ is $\phi^{-1}(\mathscr{A})$-measurable for every $\sigma \in \mathfrak{B}(K)$. Define $\hat{P}\colon X \times \mathfrak{B}(\mathbb{R}_+) \to [0, 1]$ by

$$\hat{P}(x, \sigma) = \begin{cases} \vartheta_x(\sigma \cap K) & \text{if } x \in \varOmega, \\ \delta_0(\sigma) & \text{otherwise,} \end{cases} \qquad \sigma \in \mathfrak{B}(\mathbb{R}_+).$$

It is clear that $\hat{P}$ is a $\phi^{-1}(\mathscr{A})$-measurable family of probability measures. By (47), the nth moments of the measures $P(x, \cdot)$ and $\hat{P}(x, \cdot)$ coincide for all $n \in \mathbb{Z}_+$ and for μ-a.e. $x \in X$. Hence $P(x, \cdot) = \hat{P}(x, \cdot)$ for μ-a.e. $x \in X$. This yields

$$\mathsf{E}(P(\cdot, \sigma))(x) = \mathsf{E}(\hat{P}(\cdot, \sigma))(x) = \hat{P}(x, \sigma) = P(x, \sigma)$$

for μ-a.e. $x \in X$ and for all $\sigma \in \mathfrak{B}(\mathbb{R}_+)$. This completes the proof. □

3. Applications and examples

3.1. The matrix case

Fix a positive integer κ. Denote by ω_κ the κ-dimensional Lebesgue measure on the κ-dimensional Euclidean space $\mathbb{R}^\kappa$. We begin by introducing a class of densities on $\mathbb{R}^\kappa$. Denote by $\mathscr{H}$ the set of all entire functions γ on $\mathbb{C}$ of the form

$$\gamma(z) = \sum_{n=0}^{\infty} a_n z^n, \quad z \in \mathbb{C}, \tag{48}$$

where a_n are nonnegative real numbers and $a_k > 0$ for some $k \geqslant 1$. Let γ be in $\mathscr{H}$ and $\|\cdot\|$ be a norm on $\mathbb{R}^\kappa$ induced by an inner product. Define the σ-finite Borel measure μ_γ on $\mathbb{R}^\kappa$ by $\mu_\gamma(\mathrm{d}x) = \gamma(\|x\|^2)\omega_\kappa(\mathrm{d}x)$. Given a linear transformation A of $\mathbb{R}^\kappa$, one can verify that the composition operator C_A in $L^2(\mu_\gamma)$ is well-defined if and only if A is invertible. If this is the case, then (cf. [56, Eq. (2.1)])

$$\mathsf{h}_A(x) = \frac{1}{|\det A|} \frac{\gamma(\|A^{-1}x\|^2)}{\gamma(\|x\|^2)}, \quad x \in \mathbb{R}^\kappa \setminus \{0\}. \tag{49}$$

Hence, each well-defined composition operator C_A is automatically densely defined and injective (because $0 < \mathsf{h}_A < \infty$ a.e. $[\mu_\gamma]$). We refer the reader to [56] for more information on this class of operators (see [43] for the case of Gaussian density).

The main result of this section will be preceded by an auxiliary lemma concerning the measurability of convolution powers of families of Borel measures on $\mathbb{R}_+$. Given $n \in \mathbb{N}$ and a finite Borel measure ν on $\mathbb{R}_+$, we define the nth multiplicative convolution power ν^{*n} of ν by

$$\nu^{*n}(\sigma) = \int_0^\infty \dots \int_0^\infty \chi_\sigma(t_1 \cdots t_n)\nu(\mathrm{d}t_1)\dots\nu(\mathrm{d}t_n), \quad \sigma \in \mathfrak{B}(\mathbb{R}_+). \tag{50}$$

We also set $\nu^{*0}(\sigma) = \chi_\sigma(1)$ for $\sigma \in \mathfrak{B}(\mathbb{R}_+)$. The standard measure-theoretic argument shows that for every Borel function $f\colon \mathbb{R}_+ \to \overline{\mathbb{R}}_+$,

$$\int_0^\infty f(t)\nu^{*n}(\mathrm{d}t) = \int_0^\infty \dots \int_0^\infty f(t_1 \cdots t_n)\nu(\mathrm{d}t_1)\dots\nu(\mathrm{d}t_n), \quad n \in \mathbb{N}. \tag{51}$$

Lemma 31. *Let $(X, \mathscr{A})$ be a measurable space and $\{\nu_x\colon x \in X\}$ be a family of finite Borel measures on $\mathbb{R}_+$ such that the function $X \ni x \mapsto \nu_x(\sigma) \in \mathbb{R}_+$ is $\mathscr{A}$-measurable for all $\sigma \in \mathfrak{B}(\mathbb{R}_+)$. Then the function $X \ni x \mapsto \nu_x^{*n}(\sigma)$ is $\mathscr{A}$-measurable for all $\sigma \in \mathfrak{B}(\mathbb{R}_+)$ and $n \in \mathbb{Z}_+$.*

Proof. We can assume that $n \geqslant 2$. Suppose first that there exists $R \in \mathbb{R}_+$ such that the closed support of each measure ν_x is contained in $K := [0, R]$. The standard measure-theoretic argument shows that the function $X \ni x \mapsto \int_0^\infty t^m \nu_x(\mathrm{d}t) \in \mathbb{R}_+$ is $\mathscr{A}$-measurable for all $m \in \mathbb{Z}_+$. It follows from (51) that

$$\int_0^\infty t^m \nu_x^{*n}(\mathrm{d}t) = \Big(\int_0^\infty t^m \nu_x(\mathrm{d}t)\Big)^n, \quad x \in X, m \in \mathbb{Z}_+. \tag{52}$$

By [60, Corollary 3.4], the closed support of ν_x^{*n} is contained in $[0, R^n]$ for every $x \in X$. Note that $\nu_x^{*n} = 0$ whenever $\nu_x(\mathbb{R}_+) = 0$. Since $X \ni x \mapsto \nu_x(\mathbb{R}_+) \in \mathbb{R}_+$ is $\mathscr{A}$-measurable, we deduce from (52) and Lemma 11 that the function $X \ni x \mapsto \nu_x^{*n}(\sigma) \in \mathbb{R}_+$ is $\mathscr{A}$-measurable for all $\sigma \in \mathfrak{B}(\mathbb{R}_+)$.

Coming back to the general case, we set $\nu_{k,x}(\sigma) = \nu_x(\sigma \cap [0, k])$ for $x \in X$, $\sigma \in \mathfrak{B}(\mathbb{R}_+)$ and $k \in \mathbb{N}$. Applying the above to $\{\nu_{k,x}\colon x \in X\}$, $k \in \mathbb{N}$, and using (50), we complete the proof. $\square$

Now we show that if a linear transformation A of $\mathbb{R}^\kappa$ is normal[3] in $(\mathbb{R}^\kappa, \|\cdot\|)$, then the composition operator C_A is subnormal in $L^2(\mu_\gamma)$. As shown in [56, Theorem 2.5], the converse implication is true for bounded composition operators (see also [15, Theorem 3.6] for the case of families of composition operators). It is an open question whether this is true for unbounded operators.

Theorem 32. *Let γ be in $\mathscr{H}$, $\|\cdot\|$ be a norm on $\mathbb{R}^\kappa$ induced by an inner product and A be an invertible linear transformation of $\mathbb{R}^\kappa$. If A is normal in $(\mathbb{R}^\kappa, \|\cdot\|)$, then C_A is subnormal in $L^2(\mu_\gamma)$.*

Proof. Let $(\mathbb{C}^\kappa, \|\cdot\|_{\mathrm{c}})$ be the Hilbert space complexification of $(\mathbb{R}^\kappa, \|\cdot\|)$ with the inner product $\langle\cdot, -\rangle_{\mathrm{c}}$ and A_{c} be the corresponding complexification of A. Then A_{c} is invertible and normal in $(\mathbb{C}^\kappa, \|\cdot\|_{\mathrm{c}})$. Denote by E the spectral measure of $|A_{\mathrm{c}}|^{-2}$. For $x \in \mathbb{R}^\kappa$, define the finite Borel measure ν_x on $\mathbb{R}_+$ by $\nu_x(\sigma) = \langle E(\sigma)x, x\rangle_{\mathrm{c}}$ for $\sigma \in \mathfrak{B}(\mathbb{R}_+)$. Since A_{c} is normal, we see that $A_{\mathrm{c}}E(\cdot) = E(\cdot)A_{\mathrm{c}}$, which yields

$$\nu_{Ax}(\sigma) = \langle |A_{\mathrm{c}}|^2 E(\sigma)x, x\rangle_{\mathrm{c}} = \langle (|A_{\mathrm{c}}|^{-2})^{-1} E(\sigma)x, x\rangle_{\mathrm{c}} = \int_\sigma \frac{1}{t}\nu_x(\mathrm{d}t) \tag{53}$$

for all $x \in \mathbb{R}^\kappa$ and $\sigma \in \mathfrak{B}(\mathbb{R}_+)$. Noting that the function $\mathbb{R}^\kappa \ni x \mapsto \nu_x(\sigma) \in \mathbb{R}_+$ is continuous for every $\sigma \in \mathfrak{B}(\mathbb{R}_+)$ and applying Lemma 31, we deduce that the mapping $P\colon \mathbb{R}^\kappa \times \mathfrak{B}(\mathbb{R}_+) \to [0, 1]$ given by

$$P(x, \sigma) = \begin{cases} \dfrac{1}{\gamma(\|x\|^2)} \displaystyle\sum_{n=0}^{\infty} a_n \nu_x^{*n}(|\det A| \cdot \sigma) & \text{if } x \neq 0, \\ \chi_\sigma(1) & \text{if } x = 0, \end{cases} \quad \sigma \in \mathfrak{B}(\mathbb{R}_+), \tag{54}$$

is a $\mathfrak{B}(\mathbb{R}^\kappa)$-measurable family of probability measures, where $\{a_n\}_{n=0}^\infty$ is as in (48) and $|\det A| \cdot \sigma := \{|\det A|\, t \colon t \in \sigma\}$.

We claim that P satisfies (SCC). For this, note that

$$\begin{aligned} \int_\sigma t\,\nu_{Ax}^{*n}(\mathrm{d}t) &\overset{(51)}{=} \int_0^\infty \cdots \int_0^\infty \chi_\sigma(t_1 \cdots t_n) \cdot t_1 \cdots t_n \nu_{Ax}(\mathrm{d}t_1) \dots \nu_{Ax}(\mathrm{d}t_n) \\ &\overset{(53)}{=} \int_0^\infty \cdots \int_0^\infty \chi_\sigma(t_1 \cdots t_n) \nu_x(\mathrm{d}t_1) \dots \nu_x(\mathrm{d}t_n) \\ &\overset{(50)}{=} \nu_x^{*n}(\sigma), \quad x \in \mathbb{R}^\kappa,\ \sigma \in \mathfrak{B}(\mathbb{R}_+),\ n \in \mathbb{N}. \end{aligned} \tag{55}$$

[3] Equivalently: VAV^{-1} is normal in $(\mathbb{R}^\kappa, |\cdot|)$, where $|\cdot|$ is the Euclidean norm and V is a positive invertible operator in $(\mathbb{R}^\kappa, |\cdot|)$ such that $\|x\| = |Vx|$ for all $x \in X$ (cf. [56, p. 310]).

Now, by applying the measure transport theorem and (49), we get[4]

$$
\begin{aligned}
\frac{1}{\mathsf{h}_A(A(x))}\int_\sigma t\,P(Ax,\mathrm{d}t) &\overset{(54)}{=} \frac{1}{\gamma(\|x\|^2)}\sum_{n=0}^{\infty} a_n \int_\sigma |\det A|\cdot t\,\nu_{Ax}^{*n}(|\det A|\cdot \mathrm{d}t)\\
&= \frac{1}{\gamma(\|x\|^2)}\sum_{n=0}^{\infty} a_n \int\limits_{|\det A|\cdot\sigma} t\,\nu_{Ax}^{*n}(\mathrm{d}t)\\
&\overset{(55)}{=} \frac{1}{\gamma(\|x\|^2)}\sum_{n=0}^{\infty} a_n \nu_x^{*n}(|\det A|\cdot\sigma)\\
&= P(x,\sigma), \quad x\in\mathbb{R}^\kappa\setminus\{0\}, \sigma\in\mathfrak{B}(\mathbb{R}_+),
\end{aligned}
$$

which proves our claim. Applying Theorem 9 completes the proof. □

The part of the proof of Theorem 32 regarding the $\mathfrak{B}(\mathbb{R}^\kappa)$-measurability of the family P is based on Lemma 31. Although in the matrix case this can be justified in an elementary way, Lemma 31 is much more general and fits well into the context of Lemma 11.

We conclude this section by noticing that Theorem 32 remains true for composition operators whose symbols are invertible $\mathbb{C}$-linear transformations of $\mathbb{C}^\kappa$. The proof goes along the same lines with one exception, namely we have to replace $|\det A|$ by $|\det A|^2$ (cf. [56, Section 3]).

3.2. *The discrete case*

In this section we assume that $(X,\mathscr{A},\mu)$ is a *discrete measure space*, i.e., X is a countably infinite set, $\mathscr{A}=2^X$ and μ is a σ-finite measure on $\mathscr{A}$ (or equivalently, $\mu(\{x\})<\infty$ for every $x\in X$). Let ϕ be a transformation of X. Clearly, ϕ is $\mathscr{A}$-measurable. To simplify notation, we write $\mu(x)=\mu(\{x\})$ and $\phi_\bullet^{-1}(\{x\})=\{y\in\phi^{-1}(\{x\})\colon \mu(y)>0\}$ for $x\in X$. The transformation ϕ is nonsingular if and only if $\mu(\phi^{-1}(\{x\}))=0$ for every $x\in X$ such that $\mu(x)=0$. Hence, if $\mu(x)>0$ for every $x\in X$, then ϕ is nonsingular. Assume that ϕ is nonsingular. Setting $\mathsf{h}_{\phi^n}(x)=1$ if $\mu(x)=0$, we see that

$$
\mathsf{h}_{\phi^n}(x)=\frac{\mu(\phi^{-n}(\{x\}))}{\mu(x)},\quad x\in X,\ n\in\mathbb{Z}_+. \tag{56}
$$

(Recall that, according to our convention, $\frac{0}{0}=1$.) Thus $\mathsf{h}_\phi<\infty$ a.e. $[\mu]$ if and only if $\mu(\phi^{-1}(\{x\}))<\infty$ for every $x\in X$ with $\mu(x)>0$. The positivity of h_ϕ and surjectivity of ϕ relates to each other as follows.

[4] The notation $\nu_{Ax}^{*n}(|\det A|\cdot\mathrm{d}t)$ is used when integrating with respect to the measure

$$
\mathfrak{B}(\mathbb{R}_+)\ni\sigma\mapsto\nu_{Ax}^{*n}(|\det A|\cdot\sigma)\in\mathbb{R}_+.
$$

Lemma 33. *If $\mu(x) > 0$ for all $x \in X$, then $\mathsf{h}_\phi(x) > 0$ for all $x \in X$ if and only if $\phi(X) = X$.*

Proof. Note that for every $x \in X$, $\mathsf{h}_\phi(x) > 0$ if and only if $\phi^{-1}(\{x\}) \neq \varnothing$. $\square$

Assume that $\mathsf{h}_\phi < \infty$ a.e. $[\mu]$. Since $X = \bigsqcup_{x\in\phi(X)} \phi^{-1}(\{x\})$, we get

$$\phi^{-1}(\mathscr{A}) = \Big\{ \bigsqcup_{x\in\varDelta} \phi^{-1}(\{x\}) \colon \varDelta \subseteq \phi(X) \Big\},$$

where the symbol "$\bigsqcup$" is used to denote pairwise disjoint union of sets. Note that a function f on X taking values in $\overline{\mathbb{R}}_+$ or in $\mathbb{C}$ is $\phi^{-1}(\mathscr{A})$-measurable if and only if f is constant on $\phi^{-1}(\{x\})$ for every $x \in \phi(X)$. Setting $\mathsf{E}(f) = 1$ on $\phi^{-1}(\{x\})$ if $\mu(\phi^{-1}(\{x\})) = 0$, we see that

$$\mathsf{E}(f) = \sum_{x\in\phi(X)} \frac{\int_{\phi^{-1}(\{x\})} f \,\mathrm{d}\mu}{\mu(\phi^{-1}(\{x\}))} \cdot \chi_{\phi^{-1}(\{x\})} \tag{57}$$

for every function $f\colon X \to \overline{\mathbb{R}}_+$. By linearity this equality holds a.e. $[\mu]$ for every $f \in L^2(\mu)$ as well.

Now we investigate the consistency condition (CC) in the context of discrete measure spaces. Since $\mathscr{A} = 2^X$, we can abbreviate the expression "an $\mathscr{A}$-measurable family of probability measures" to "a family of probability measures".

Lemma 34. *Let $(X, \mathscr{A}, \mu)$ be a discrete measure space, $P\colon X \times \mathfrak{B}(\mathbb{R}_+) \to [0, 1]$ be a family of probability measures and ϕ be a nonsingular transformation of X such that $\mathsf{h}_\phi < \infty$ a.e. $[\mu]$. Then (CC) is equivalent to each of the following conditions*:

(i) *for every $x \in X$ such that $\mu(\phi^{-1}(\{x\})) > 0$, the following holds*:

$$\int_\sigma t P(x, \mathrm{d}t) = \sum_{y\in\phi_\bullet^{-1}(\{x\})} \frac{\mu(y)}{\mu(x)} P(y, \sigma), \quad \sigma \in \mathfrak{B}(\mathbb{R}_+),$$

(ii) *for every $x \in X$ such that $\mu(\phi^{-1}(\{x\})) > 0$, the following hold*:

$$P(y, \{0\}) = 0 \text{ for every } y \in \phi_\bullet^{-1}(\{x\}), \text{ and}$$

$$P(x, \sigma) = \sum_{y\in\phi_\bullet^{-1}(\{x\})} \frac{\mu(y)}{\mu(x)} \int_\sigma \frac{1}{t} P(y, \mathrm{d}t), \quad \sigma \in \mathfrak{B}(\mathbb{R}_+),$$

(iii) *for every* $x \in X$ *such that* $\mu(\phi^{-1}(\{x\})) > 0$, *the following hold*:

$$P(y,\cdot) \ll P(x,\cdot) \text{ for every } y \in \phi_{\bullet}^{-1}(\{x\}), \text{ and}$$
$$t = \sum_{y \in \phi_{\bullet}^{-1}(\{x\})} \frac{\mu(y)}{\mu(x)} \cdot \frac{\mathrm{d}P(y,\cdot)}{\mathrm{d}P(x,\cdot)}(t) \text{ for } P(x,\cdot)\text{-a.e. } t \in \mathbb{R}_+,$$

(iv) *for every* $x \in X$ *such that* $\mu(\phi^{-1}(\{x\})) > 0$, *the following hold*:

$$P(x,\{0\}) = 0, P(y,\cdot) \ll P(x,\cdot) \text{ for every } y \in \phi_{\bullet}^{-1}(\{x\}), \text{ and}$$
$$1 = \sum_{y \in \phi_{\bullet}^{-1}(\{x\})} \frac{\mu(y)}{\mu(x)} \cdot \frac{1}{t} \cdot \frac{\mathrm{d}P(y,\cdot)}{\mathrm{d}P(x,\cdot)}(t) \text{ for } P(x,\cdot)\text{-a.e. } t \in \mathbb{R}_+.$$

Proof. Applying (56), (57) and the decomposition $X = \bigsqcup_{x \in \phi(X)} \phi^{-1}(\{x\})$, we deduce that (CC) is equivalent to (i). In turn, employing (56), (57) and Lemma 14(ii), we verify that (i) is equivalent to (ii). By the Radon–Nikodym theorem, (i) is easily seen to be equivalent to (iii).

(ii)⇒(iv) Since (ii) implies (iii), it suffices to show that $P(x,\{0\}) = 0$ whenever $\mu(\phi^{-1}(\{x\})) > 0$. Suppose that, on the contrary, there exists $x \in X$ such that $\mu(\phi^{-1}(\{x\})) > 0$ and $P(x,\{0\}) > 0$. Since ϕ is nonsingular, we see that $\mu(x) > 0$. Hence $x \in \phi_{\bullet}^{-1}(\{\phi(x)\})$, and thus by (ii) $P(x,\{0\}) = 0$, a contradiction.

(iv)⇒(iii) Evident. □

The above preparation enables us to state a discrete version of Theorem 9.

Theorem 35. *Let* $(X, \mathscr{A}, \mu)$ *be a discrete measure space and* ϕ *be a transformation of* X *such that*

(i) *for every* $x \in X$, $\mu(x) = 0$ *if and only if* $\mu(\phi^{-1}(\{x\})) = 0$,
(ii) $\mu(\phi^{-1}(\{x\})) < \infty$ *for every* $x \in X$ *such that* $\mu(x) > 0$.

Suppose there exists a family $P\colon X \times \mathfrak{B}(\mathbb{R}_+) \to [0,1]$ *of probability measures which satisfies one of the equivalent conditions* (i) *to* (iv) *of Lemma* 34. *Then* C_ϕ *is subnormal.*

Proof. By (56), the conjunction of the conditions (i) and (ii) is equivalent to requiring that ϕ is nonsingular and $0 < \mathsf{h}_\phi < \infty$ a.e. $[\mu]$. Combining [13, Proposition 6.2] and (4), we see that $0 < \mathsf{h}_\phi < \infty$ a.e. $[\mu]$ if and only if C_ϕ is injective and densely defined. Hence, by applying Lemma 34 and Theorem 9 with $\zeta(t) = t$, we complete the proof. □

It is worth mentioning that if ϕ is an injective nonsingular transformation of a discrete measure space, then, by (56), $\mathsf{h}_{\phi^n} < \infty$ a.e. $[\mu]$ for every $n \in \mathbb{N}$, and thus, by [13, Corollary 4.5 and Theorem 4.7], $\mathscr{D}^\infty(C_\phi)$ is a core for C_ϕ^n for every $n \in \mathbb{Z}_+$. Moreover, the conditional expectation $\mathsf{E}(\cdot)$ acts as the identity map (see the paragraph just below (44)). Hence (CC) becomes (SCC). This observation enables us to apply the results of Section 2.4. In particular, combining Propositions 24(i) and 26(v), we get the following.

Proposition 36. *Let $(X, \mathscr{A}, \mu)$ be a discrete measure space and ϕ be an injective nonsingular transformation of X. Assume that $P: X \times \mathfrak{B}(\mathbb{R}_+) \to [0,1]$ is a family of probability measures which satisfies (CC). Then*

(i) $\int_\sigma t^n P(\phi^n(x), \mathrm{d}t) = \mathsf{h}_{\phi^n}(\phi^n(x)) \cdot P(x, \sigma)$ *for all* $\sigma \in \mathfrak{B}(\mathbb{R}_+)$, $n \in \mathbb{Z}_+$ *and* $x \in X$ *such that* $\mu(x) > 0$.

Moreover, if $\mu(x) > 0$ for every $x \in X$, then

(ii) $\int_\sigma t^n P(x, \mathrm{d}t) = \mathsf{h}_{\phi^n}(x) \cdot P\big((\phi^n)^{-1}(x), \sigma\big)$ *for all* $\sigma \in \mathfrak{B}(\mathbb{R}_+)$, $x \in \phi^n(X)$ *and* $n \in \mathbb{Z}_+$.

Below we will discuss the question of subnormality of composition operators in L^2-spaces over discrete measure spaces with injective symbols. This is done by exploiting a model for such operators which is based on [66, Proposition 2.4].

Remark 37. Suppose $(X, \mathscr{A}, \mu)$ is a σ-finite measure space such that X is at most countable, $\mathscr{A} = 2^X$ and $\mu(x) > 0$ for every $x \in X$. Let ϕ be an injective transformation of X. We say that C_ϕ is of *type* I if there exists $u \in X$ such that the mapping $\mathbb{Z}_+ \ni n \to \phi^n(u) \in X$ is bijective, of *type* II if ϕ is bijective and there exists $u \in X$ such that the mapping $\mathbb{Z} \ni n \to \phi^n(u) \in X$ is bijective, and of *type* III if there exist $u \in X$ and $m \in \mathbb{N}$ such that the mapping $\{0, \ldots, m-1\} \ni n \mapsto \phi^n(u) \in X$ is bijective (note that then $\phi^m = \mathrm{id}_X$). One can show that a composition operator of type I cannot be subnormal (in fact, it is not hyponormal because $C_\phi \chi_{\{u\}} = 0$ and $C_\phi^* \chi_{\{u\}} \neq 0$), and it is unitarily equivalent[5] to the adjoint of an injective unilateral weighted shift. A composition operator of type II is unitarily equivalent to an injective bilateral weighted shift. Hence, by applying Theorem 35, we obtain the Berger–Gellar–Wallen characterization of subnormality of injective bilateral weighted shifts (see [12, Theorem 3.2] and note that Theorem 47 follows from Theorem 35). In turn, a composition operator of type III is a bounded mth root of I (because $\dim L^2(\mu) < \infty$ and $\phi^m = \mathrm{id}_X$). Hence, by Proposition A.3, it is subnormal if and only if it is unitary. The latter happens if and only if $\mathsf{h}_\phi = 1$ (again because $\dim L^2(\mu) < \infty$), or equivalently if and only if $X \ni x \mapsto \mu(x) \in (0, \infty)$ is a constant function. It follows from [66, Proposition 2.4] and Proposition C.1 that if ϕ is an arbitrary injective transformation of X, then there exist $N \in \mathbb{N} \cup \{\infty\}$ and a sequence $\{Y_n\}_{n=1}^N \subseteq \mathscr{A}(\phi)$ of pairwise disjoint nonempty sets such that $X = \bigcup_{n=1}^N Y_n$, each $C_{\phi_{Y_n}}$ is of one of the types I, II or III, and C_ϕ is unitarily equivalent to $\bigoplus_{n=1}^N C_{\phi_{Y_n}}$

[5] Via the unitary isomorphism $U: \ell^2(\mathbb{Z}_+) \to L^2(\mu)$ given by $(Uf)(\phi^n(u)) = \frac{f(n)}{\sqrt{\mu(\phi^n(u))}}$ for $n \in \mathbb{Z}_+$ and $f \in \ell^2(\mathbb{Z}_+)$; see also [38, Remark 3.1.4].

(with the notation as in Appendix C). In view of the above discussion, if C_ϕ is subnormal then there is no summand of type I in the decomposition $\bigoplus_{n=1}^N C_{\phi_{Y_n}}$, and thus C_ϕ is unitarily equivalent to an orthogonal sum of at most countably many operators, each of which is either a subnormal injective bilateral weighted shift or a unitary mth root ($m \geqslant 1$) of the identity operator on a finite dimensional space. On the other hand, by Corollary C.2, an orthogonal sum of at most countably many composition operators of one of the types I, II or III is unitarily equivalent to a composition operator C_ϕ in an L^2-space over a σ-finite measure space $(X, 2^X, \mu)$ such that X is at most countable, $\mu(x) > 0$ for every $x \in X$ and ϕ is injective.

3.3. Local consistency

In this section we show that the "local consistency technique" introduced in [11, Lemma 4.1.3] for weighted shifts on directed trees can be implemented in the context of composition operators in L^2-spaces over discrete measure spaces. The non-discrete case does not seem to make sense. In what follows we preserve the notation from Section 3.2.

Lemma 38. *Let $(X, \mathscr{A}, \mu)$ be a discrete measure space and ϕ be a nonsingular transformation of X such that $\mathsf{h}_\phi < \infty$ a.e. $[\mu]$. Let $x \in X$ be such that $\mu(\phi^{-1}(\{x\})) > 0$ and for every $y \in \phi_\bullet^{-1}(\{x\})$, $\{\mathsf{h}_{\phi^n}(y)\}_{n=0}^\infty$ is a Stieltjes moment sequence with a representing measure ϑ_y. Then the following assertions are valid.*

(i) *If*

$$\sum_{y \in \phi_\bullet^{-1}(\{x\})} \frac{\mu(y)}{\mu(x)} \int_0^\infty \frac{1}{t} \vartheta_y(\mathrm{d}t) \leqslant 1, \tag{58}$$

then $\{\mathsf{h}_{\phi^n}(x)\}_{n=0}^\infty$ is a Stieltjes moment sequence with a representing measure $\widetilde{\vartheta}_x$ given by

$$\widetilde{\vartheta}_x(\sigma) = \sum_{y \in \phi_\bullet^{-1}(\{x\})} \frac{\mu(y)}{\mu(x)} \int_\sigma \frac{1}{t} \vartheta_y(\mathrm{d}t) + \varepsilon_x \cdot \delta_0(\sigma), \quad \sigma \in \mathfrak{B}(\mathbb{R}_+), \tag{59}$$

where

$$\varepsilon_x = 1 - \sum_{y \in \phi_\bullet^{-1}(\{x\})} \frac{\mu(y)}{\mu(x)} \int_0^\infty \frac{1}{t} \vartheta_y(\mathrm{d}t). \tag{60}$$

(ii) *If $\{\mathsf{h}_{\phi^n}(x)\}_{n=0}^\infty$ is a Stieltjes moment sequence, and $\{\mathsf{h}_{\phi^{n+1}}(x)\}_{n=0}^\infty$ is a determinate Stieltjes moment sequence, then (58) holds, the Stieltjes moment sequence $\{\mathsf{h}_{\phi^n}(x)\}_{n=0}^\infty$ is determinate and its unique representing measure $\widetilde{\vartheta}_x$ is given by (59) and (60).*

Proof. It follows from Lemma 15 that

$$\mathsf{h}_{\phi^{n+1}}(x) = \mathsf{h}_{\phi^{n+1}}(\phi(y)) \overset{(36)}{=} \mathsf{h}_{\phi}(x) \cdot \mathsf{E}(\mathsf{h}_{\phi^n})(y), \quad y \in \phi_{\bullet}^{-1}(\{x\}), \, n \in \mathbb{Z}_+. \tag{61}$$

Using (57), we see that for every function $f: X \to \overline{\mathbb{R}}_+$,

$$(\mathsf{E}(f))(z) = \sum_{y \in \phi_{\bullet}^{-1}(\{x\})} \frac{\mu(y)}{\mu(\phi^{-1}(\{x\}))} f(y), \quad z \in \phi^{-1}(\{x\}).$$

This and (61) yield

$$\mathsf{h}_{\phi^{n+1}}(x) = \frac{\mathsf{h}_{\phi}(x)}{\mu(\phi^{-1}(\{x\}))} \sum_{y \in \phi_{\bullet}^{-1}(\{x\})} \mu(y) \int_0^\infty t^n \vartheta_y(\mathrm{d}t) = \int_0^\infty t^n \nu_x(\mathrm{d}t), \quad n \in \mathbb{Z}_+,$$

where ν_x is the Borel measure on $\mathbb{R}_+$ given by

$$\nu_x = \frac{\mathsf{h}_{\phi}(x)}{\mu(\phi^{-1}(\{x\}))} \sum_{y \in \phi_{\bullet}^{-1}(\{x\})} \mu(y) \cdot \vartheta_y.$$

Hence, $\{\mathsf{h}_{\phi^{n+1}}(x)\}_{n=0}^\infty$ is a Stieltjes moment sequence with the representing measure ν_x. Noticing that $\mathsf{h}_{\phi^0}(x) = 1$ and

$$\int_\sigma \frac{1}{t} \nu_x(\mathrm{d}t) = \sum_{y \in \phi_{\bullet}^{-1}(\{x\})} \frac{\mu(y)}{\mu(x)} \int_\sigma \frac{1}{t} \vartheta_y(\mathrm{d}t), \quad \sigma \in \mathfrak{B}(\mathbb{R}_+),$$

we can apply [11, Lemma 2.4.1] with $\vartheta = 1$ to obtain (i) and (ii). This completes the proof. $\square$

Remark 39. Regarding Lemma 38, it is worth pointing out that if $\mathsf{E}(\mathsf{h}_{\phi^n}) = \mathsf{h}_{\phi^n}$ a.e. $[\mu]$ for every $n \in \mathbb{Z}_+$, then assertions (i) and (ii) are still valid if (58) is replaced by

$$\mathsf{h}_{\phi}(x) \int_0^\infty \frac{1}{t} \vartheta_y(\mathrm{d}t) \leqslant 1 \text{ for some } y \in \phi_{\bullet}^{-1}(\{x\}),$$

and (59) and (60) are replaced by (with the above y)

$$\widetilde{\vartheta}_x(\sigma) = \mathsf{h}_{\phi}(x) \int_\sigma \frac{1}{t} \vartheta_y(\mathrm{d}t) + \varepsilon_x \cdot \delta_0(\sigma) \text{ with } \varepsilon_x = 1 - \mathsf{h}_{\phi}(x) \int_0^\infty \frac{1}{t} \vartheta_y(\mathrm{d}t).$$

Indeed, in view of (61), the Stieltjes moment sequence $\{\mathsf{h}_{\phi^{n+1}}(x)\}_{n=0}^\infty$ is represented by the measure $\mathsf{h}_{\phi}(x) \cdot \vartheta_y$ and thus we can apply [11, Lemma 2.4.1]. Note that under the circumstances of (ii) the measure ϑ_y does not depend on $y \in \phi_{\bullet}^{-1}(\{x\})$.

It is worth mentioning that Lemma 38 does not exclude the possibility that the transformation ϕ has an essential fixed point x, i.e., $x \in \phi_{\bullet}^{-1}(\{x\})$ (under the assumption $\mu(\phi^{-1}(\{x\})) > 0$, this is equivalent to $\phi(x) = x$). We will show that if this is the case (cf. Example 42), then, under the determinacy assumption, all the representing measures ϑ_y are concentrated on the interval $(1, \infty)$ except for ϑ_x which is concentrated on $[1, \infty)$.

Lemma 40. *Under the assumptions of Lemma 38, if $\{\mathsf{h}_{\phi^n}(x)\}_{n=0}^{\infty}$ is a Stieltjes moment sequence, $\{\mathsf{h}_{\phi^{n+1}}(x)\}_{n=0}^{\infty}$ is a determinate Stieltjes moment sequence and $x \in \phi_{\bullet}^{-1}(\{x\})$, then $\vartheta_x([0,1)) = 0$ and $\vartheta_y([0,1]) = 0$ for every $y \in \phi_{\bullet}^{-1}(\{x\}) \setminus \{x\}$.*

Proof. Since, by Lemma 38(ii), the sequence $\{\mathsf{h}_{\phi^n}(x)\}_{n=0}^{\infty}$ is determinate, we deduce that $\widetilde{\vartheta}_x = \vartheta_x$ (with $\widetilde{\vartheta}_x$ as in Lemma 38). By (58), $\vartheta_y(\{0\}) = 0$ for all $y \in \phi_{\bullet}^{-1}(\{x\})$. In view of (59), we see that for every $\sigma \in \mathfrak{B}(\mathbb{R}_+)$,

$$\int_{\sigma} \Big(1 - \frac{1}{t}\Big) \vartheta_x(\mathrm{d}t) = \sum_{y \in \phi_{\bullet}^{-1}(\{x\}) \setminus \{x\}} \frac{\mu(y)}{\mu(x)} \int_{\sigma} \frac{1}{t} \vartheta_y(\mathrm{d}t) + \varepsilon_x \cdot \delta_0(\sigma), \tag{62}$$

with the convention that $\sum_{y \in \varnothing} v_y = 0$. Since the right-hand side of the equality in (62) is nonnegative, we conclude that the measure ϑ_x is concentrated on $[1, \infty)$. Hence, $\varepsilon_x = 0$ and each measure ϑ_y, $y \in \phi_{\bullet}^{-1}(\{x\})$, is concentrated on $[1, \infty)$. Substituting $\sigma = \{1\}$ into (62) completes the proof. $\square$

The local consistency technique enables us to prove the subnormality of injective composition operators C_ϕ under certain determinacy assumption. Theorem 41 below can be regarded as a counterpart of [11, Theorem 5.1.3].

Theorem 41. *Let $(X, \mathscr{A}, \mu)$ be a discrete measure space and ϕ be a nonsingular transformation of X such that $\{\mathsf{h}_{\phi^n}(x)\}_{n=0}^{\infty}$ is a Stieltjes moment sequence and $\{\mathsf{h}_{\phi^{n+1}}(x)\}_{n=0}^{\infty}$ is a determinate Stieltjes moment sequence for μ-a.e. $x \in X$. Then C_ϕ is subnormal if and only if $\mathsf{h}_\phi > 0$ a.e. $[\mu]$. In particular, C_ϕ is subnormal if $\mu(x) > 0$ for every $x \in X$.*

Proof. Suppose $\mathsf{h}_\phi > 0$ a.e. $[\mu]$. Set $X_{\bullet} = \{x \in X \colon \mu(x) > 0\}$. We infer from (56) that $X_{\bullet} = \{x \in X \colon \mu(\phi^{-1}(\{x\})) > 0\}$. By Lemma 38(ii), for every $x \in X_{\bullet}$, the Stieltjes moment sequence $\{\mathsf{h}_{\phi^n}(x)\}_{n=0}^{\infty}$ is determinate; denote its unique representing measure by $P(x, \cdot)$. Set $P(x, \cdot) = \delta_0$ for $x \in X \setminus X_{\bullet}$. Since $\mathsf{h}_{\phi^0} \equiv 1$, we see that $P \colon X \times \mathfrak{B}(\mathbb{R}_+) \to [0,1]$ is a family of probability measures. By Lemma 38(ii), we have

$$P(x, \sigma) = \sum_{y \in \phi_{\bullet}^{-1}(\{x\})} \frac{\mu(y)}{\mu(x)} \int_{\sigma} \frac{1}{t} P(y, \mathrm{d}t) + \varepsilon_x \cdot \delta_0(\sigma), \quad \sigma \in \mathfrak{B}(\mathbb{R}_+),\, x \in X_{\bullet}. \tag{63}$$

It follows from (63) that $P(y, \{0\}) = 0$ for all $y \in \phi_{\bullet}^{-1}(\{x\})$ and $x \in X_{\bullet}$. Since $x \in \phi^{-1}(\{\phi(x)\})$ for every $x \in X$, we deduce that $\phi(x) \in X_{\bullet}$ and $x \in \phi_{\bullet}^{-1}(\{\phi(x)\})$ for every

$x \in X_\bullet$. Hence $P(x, \{0\}) = 0$ for every $x \in X_\bullet$. Substituting $\sigma = \{0\}$ into (63), we deduce that $\varepsilon_x = 0$ for every $x \in X_\bullet$. This means that condition (ii) of Lemma 34 is satisfied. By Theorem 35, C_ϕ is subnormal. The reverse implication follows from [13, Proposition 6.2 and Corollary 6.3].

Suppose now that $\mu(x) > 0$ for every $x \in X$. Note that for every $x \in X$, the Stieltjes moment sequence $\{\mathsf{h}_{\phi^n}(x)\}_{n=0}^\infty$ is determinate (see e.g., [11, Lemma 2.4.1]); denote its representing measure by ϑ_x. In view of the previous paragraph and Lemma 33, it suffices to show that $\phi(X) = X$. Suppose that, contrary to our claim, there exists $x_0 \in X \setminus \phi(X)$. Then $\phi^{-n}(\{x_0\}) = \varnothing$ for all $n \geqslant 1$, which implies that $\vartheta_{x_0} = \delta_0$. Observe that $x_0 \in \phi_\bullet^{-1}(\{\phi(x_0)\})$. Applying Lemma 38(ii) to $x = \phi(x_0)$ and using (58), we deduce that $\vartheta_{x_0}(\{0\}) = 0$, which contradicts $\vartheta_{x_0} = \delta_0$. This completes the proof. $\square$

3.4. A single essential fixed point

Now we address the question of subnormality of composition operators induced by a transformation which has a single essential fixed point x, i.e., $\phi^{-1}(\{x\})$ is a two-point set and $\phi^{-1}(\{y\})$ is a one-point set for every $y \neq x$. The situation seems to be simple, but it is not. It leads to nontrivial questions in the theory of moment problems. This enables us to construct unbounded subnormal composition operators C_ϕ with the sequence $\{\mathsf{h}_{\phi^{n+1}}(0)\}_{n=0}^\infty$ being either determinate or indeterminate according to our needs. For them the equalities $\mathsf{E}(\mathsf{h}_{\phi^n}) = \mathsf{h}_{\phi^n}$ a.e. $[\mu]$, $n \in \mathbb{Z}_+$, cannot hold. This is also rare in the bounded case.

Example 42. Let $(X, \mathscr{A}, \mu)$ be a discrete measure space with $X = \mathbb{Z}_+$ such that $\mu(n) > 0$ for every $n \in \mathbb{Z}_+$. Assume that $\mu(0) = 1$. Define the (nonsingular) transformation ϕ of $\mathbb{Z}_+$ by $\phi(0) = 0$ and $\phi(n) = n - 1$ for $n \geqslant 1$. By (56), we have

$$\mathsf{h}_{\phi^n}(k) = \begin{cases} \dfrac{\mu(n+k)}{\mu(k)} & \text{if } k \geqslant 1, \\ \displaystyle\sum_{j=0}^{n} \mu(j) & \text{if } k = 0, \end{cases} \qquad n \in \mathbb{Z}_+. \tag{64}$$

Since $\{\chi_{\{x\}} : x \in X\} \subseteq \mathcal{D}^\infty(C_\phi)$, we see that $\mathcal{D}^\infty(C_\phi)$ is dense in $L^2(\mu)$.

Suppose $\{\mathsf{h}_{\phi^n}(0)\}_{n=0}^\infty$ is a Stieltjes moment sequence with a representing measure ϑ_0, $\{\mathsf{h}_{\phi^{n+1}}(0)\}_{n=0}^\infty$ is a determinate Stieltjes moment sequence and $\{\mathsf{h}_{\phi^n}(1)\}_{n=0}^\infty$ is a Stieltjes moment sequence with a representing measure ϑ_1. It follows from Lemma 40, applied to $x = 0$, that $\vartheta_0([0, 1)) = \vartheta_1([0, 1]) = 0$. We claim that the Stieltjes moment sequence $\{\mathsf{h}_{\phi^n}(1)\}_{n=0}^\infty$ is determinate,

$$\int_0^\infty \frac{\mu(1)}{t-1} \vartheta_1(\mathrm{d}t) \leqslant 1$$

and

$$\vartheta_0(\sigma) = \int\limits_{\sigma} \frac{\mu(1)}{t-1}\vartheta_1(\mathrm{d}t) + \varepsilon\delta_1(\sigma), \quad \sigma \in \mathfrak{B}(\mathbb{R}_+),$$

with

$$\varepsilon = 1 - \int\limits_0^\infty \frac{\mu(1)}{t-1}\vartheta_1(\mathrm{d}t).$$

Indeed, by (64), we have

$$\mathsf{h}_{\phi^n}(0) = 1 + \mu(1)\int\limits_0^\infty (1 + \ldots + t^{n-1})\vartheta_1(\mathrm{d}t), \quad n \in \mathbb{N}.$$

This yields

$$\int\limits_0^\infty t^n(t-1)\vartheta_0(\mathrm{d}t) = \mathsf{h}_{\phi^{n+1}}(0) - \mathsf{h}_{\phi^n}(0) = \mu(1)\int\limits_0^\infty t^n\vartheta_1(\mathrm{d}t), \quad n \in \mathbb{Z}_+. \tag{65}$$

Note that the measure $(t-1)\vartheta_0(\mathrm{d}t)$ is determinate. Indeed, since the measure $t\vartheta_0(\mathrm{d}t)$, being a representing measure of $\{\mathsf{h}_{\phi^{n+1}}(0)\}_{n=0}^\infty$, is determinate, we infer from (1) that $\mathbb{C}[t]$ is dense in $L^2((1+t^2)t\vartheta_0(\mathrm{d}t))$. Hence, if $\sigma \in \mathfrak{B}(\mathbb{R}_+)$, then there exists a sequence $\{p_n\}_{n=1}^\infty \subseteq \mathbb{C}[t]$ such that

$$\lim_{n\to\infty}\int\limits_0^\infty |\chi_\sigma(t) - p_n(t)|^2(1+t^2)t\vartheta_0(\mathrm{d}t) = 0.$$

Therefore

$$\lim_{n\to\infty}\int\limits_0^\infty |\chi_\sigma(t) - p_n(t)|^2(1+t^2)(t-1)\vartheta_0(\mathrm{d}t) = 0.$$

This implies that $\mathbb{C}[t]$ is dense in $L^2((1+t^2)(t-1)\vartheta_0(\mathrm{d}t))$. Applying (1) again completes the proof of the determinacy of $(t-1)\vartheta_0(\mathrm{d}t)$ (because $\vartheta_0([0,1)) = 0$). This combined with (65) implies that $\{\mathsf{h}_{\phi^n}(1)\}_{n=0}^\infty$ is determinate and $\mu(1)\vartheta_1(\sigma) = \int_\sigma (t-1)\vartheta_0(\mathrm{d}t)$ for every $\sigma \in \mathfrak{B}(\mathbb{R}_+)$. Hence, for every $\sigma \in \mathfrak{B}(\mathbb{R}_+)$,

$$\vartheta_0(\sigma) = \vartheta_0(\sigma \cap (1,\infty)) + \vartheta_0(\sigma \cap \{1\}) = \int\limits_{\sigma\cap(1,\infty)} \frac{\mu(1)}{t-1}\vartheta_1(\mathrm{d}t) + \vartheta_0(\{1\})\delta_1(\sigma),$$

and $\vartheta_0(\{1\}) = \varepsilon$, which proves our claim.

The above reasoning can be reversed in a sense. Namely, we will provide a general procedure of constructing the measure μ that guarantees the subnormality of C_ϕ (with X, $\mathscr{A}$ and ϕ as at the beginning of this example and $\mu(0) = 1$). Take a Borel probability measure ϑ on $\mathbb{R}_+$ such that

$$\vartheta([0,1]) = 0,\ \ \alpha := \int_0^\infty \frac{1}{t-1}\vartheta(\mathrm{d}t) < \infty,\ \ \int_0^\infty t^n \vartheta(\mathrm{d}t) < \infty \text{ for all } n \in \mathbb{Z}_+. \tag{66}$$

Note that $\alpha > 0$. Take $\mu(1) \in (0, 1/\alpha]$ and set

$$\mu(n) = \mu(1) \int_0^\infty t^{n-1} \vartheta(\mathrm{d}t), \quad n \geqslant 2. \tag{67}$$

Clearly, $\mu(k) > 0$ for all $k \in \mathbb{Z}_+$. Define the family $P\colon \mathbb{Z}_+ \times \mathfrak{B}(\mathbb{R}_+) \to [0,1]$ of probability measures by

$$P(k,\sigma) = \begin{cases} \dfrac{\mu(1)}{\mu(k)} \displaystyle\int_\sigma t^{k-1}\vartheta(\mathrm{d}t) & \text{if } k \geqslant 1, \\ \\ \displaystyle\int_\sigma \frac{\mu(1)}{t-1}\vartheta(\mathrm{d}t) + \varepsilon\delta_1(\sigma) & \text{if } k = 0, \end{cases} \qquad \sigma \in \mathfrak{B}(\mathbb{R}_+), \tag{68}$$

with $\varepsilon = 1 - \int_0^\infty \frac{\mu(1)}{t-1}\vartheta(\mathrm{d}t)$. Observe that P satisfies condition (i) of Lemma 34. Indeed, if $k \geqslant 1$, then

$$\int_\sigma tP(k,\,\mathrm{d}t) = \frac{\mu(1)}{\mu(k)} \int_\sigma t^k \vartheta(\mathrm{d}t) = \frac{\mu(k+1)}{\mu(k)} P(k+1,\sigma), \quad \sigma \in \mathfrak{B}(\mathbb{R}_+),$$

while for $k = 0$, we have

$$\begin{aligned} \int_\sigma tP(0,\,\mathrm{d}t) &= \mu(1) \int_\sigma \frac{t}{t-1}\vartheta(\mathrm{d}t) + \varepsilon\delta_1(\sigma) \\ &= \mu(1)\vartheta(\sigma) + \mu(1)\int_\sigma \frac{1}{t-1}\vartheta(\mathrm{d}t) + \varepsilon\delta_1(\sigma) \\ &= \mu(1)P(1,\sigma) + P(0,\sigma), \quad \sigma \in \mathfrak{B}(\mathbb{R}_+). \end{aligned}$$

Hence, by Theorem 35, C_ϕ is subnormal. In view of Lemma 34 and Theorem 17, $P(k,\cdot)$ is a representing measure of $\{\mathsf{h}_{\phi^n}(k)\}_{n=0}^\infty$ for every $k \in \mathbb{Z}_+$. Note also that

$$\mathsf{E}(\mathsf{h}_{\phi^n}) = \mathsf{h}_{\phi^n} \text{ a.e. } [\mu] \text{ for all } n \in \mathbb{Z}_+ \text{ if and only if } \vartheta = \delta_{1+\mu(1)}. \tag{69}$$

(Of course, if $\vartheta = \delta_{1+\mu(1)}$, then $\varepsilon = 0$.) Indeed, it is clear that $\mathsf{E}(\mathsf{h}_{\phi^n}) = \mathsf{h}_{\phi^n}$ a.e. $[\mu]$ for all $n \in \mathbb{Z}_+$ if and only if $\mathsf{h}_{\phi^n}(0) = \mathsf{h}_{\phi^n}(1)$ for all $n \in \mathbb{Z}_+$ (cf. (57)), or equivalently if and only if $\sum_{j=0}^{n} \mu(j) = \mu(n+1)/\mu(1)$ for all $n \in \mathbb{Z}_+$ (cf. (64)). By induction on n, the latter holds if and only if $\mu(n+1) = \mu(1)(1+\mu(1))^n$ for all $n \in \mathbb{Z}_+$. This and (67) (consult also (1)) completes the proof of (69). We point out that the situation described in (69) may happen only when $C_\phi \in \boldsymbol{B}(L^2(\mu))$, and if this is the case, then $\|C_\phi\|^2 = 1 + \mu(1)$ (cf. (70)).

Note that if ϑ and μ are as in (66) and (67) with $\vartheta(\mathbb{R}_+) = 1$, $\mu(0) = 1$ and $\mu(1) \in (0, 1/\alpha]$, then $C_\phi \in \boldsymbol{B}(L^2(\mu))$ if and only if $\sup(\operatorname{supp}\vartheta) < \infty$. Indeed, by (64), $C_\phi \in \boldsymbol{B}(L^2(\mu))$ if and only if $\beta < \infty$, where $\beta := \sup_{k\geqslant 1} \frac{\mu(k+1)}{\mu(k)}$. Since $\vartheta(\mathbb{R}_+) = 1$, we infer from (67) that $\{\mu(k+1)\}_{k=0}^{\infty}$ is a Stieltjes moment sequence with a representing measure $\mu(1)\vartheta(\mathrm{d}t)$. Hence, by Lemma 2, we see that $\beta < \infty$ if and only if $\sup(\operatorname{supp}\vartheta) < \infty$. Moreover, if $C_\phi \in \boldsymbol{B}(L^2(\mu))$, then by Lemma 2, [46, Theorem 1] and (64) we have

$$\|C_\phi\|^2 = \max\big\{1 + \mu(1), \sup(\operatorname{supp}\vartheta)\big\}. \tag{70}$$

Now we provide explicit examples of measures ϑ leading to unbounded subnormal C_ϕ's for which the sequence $\{\mathsf{h}_{\phi^{n+1}}(0)\}_{n=0}^{\infty}$ is either determinate or indeterminate according to our needs. We begin with the determinate case. Set

$$\vartheta = c^{-1} \sum_{j=2}^{\infty} j^{-1} \mathrm{e}^{-j^2} \delta_j \text{ and } \gamma_n = \int_0^\infty t^n \vartheta(\mathrm{d}t) \text{ for } n \in \mathbb{Z}_+,$$

where $c = \sum_{j=2}^{\infty} j^{-1} \mathrm{e}^{-j^2}$. It is easily seen that ϑ is a probability measure which satisfies (66). Let α, μ and P be as in (66), (67) and (68) with $\mu(0) = 1$ and $\mu(1) \in (0, 1/\alpha]$. Note that there exists a positive real number b such that $\gamma_n \leqslant b n^n$ for all $n \geqslant 1$ (see [39, Example 4.2.2] and [60, Example 7.1]). This implies that there exists a positive real number b' such that $\mathsf{h}_{\phi^n}(0) = \int_0^\infty t^n P(0, \mathrm{d}t) \leqslant b' n^n$ for all $n \geqslant 1$. By the Carleman criterion (see e.g., [52, Corollary 4.5]), the Stieltjes moment sequences $\{\mathsf{h}_{\phi^n}(0)\}_{n=0}^{\infty}$ and $\{\mathsf{h}_{\phi^{n+1}}(0)\}_{n=0}^{\infty}$ are determinate.

The indeterminate case can be done as follows. Let ϑ be an indeterminate probability measure such that[6] $\vartheta([0, 2)) = 0$. Clearly ϑ satisfies (66). Set $\mu(1) = \frac{1}{\alpha}$. Then $\varepsilon = 0$ and for every Borel function $f: \mathbb{R}_+ \to \overline{\mathbb{R}}_+$,

$$\mu(1) \int_0^\infty f(t)(1 + t^2)\vartheta(\mathrm{d}t) \leqslant \int_0^\infty f(t)(1 + t^2) t P(0, \mathrm{d}t).$$

[6] Consider e.g., the measure ϑ given by $\vartheta(\sigma) = \widetilde{\vartheta}(\frac{1}{2} \cdot \sigma)$ for $\sigma \in \mathfrak{B}(\mathbb{R}_+)$, where $\widetilde{\vartheta}$ is the q-orthogonality probability measure for the Al-Salam–Carlitz polynomials ($0 < q < 1$), which is indeterminate and supported in $\{q^{-n}\}_{n=0}^{\infty}$ (cf. [20]).

By (1) and the indeterminacy of ϑ, this implies that the measure $tP(0, \mathrm{d}t)$ is indeterminate, and thus the corresponding sequence of moments $\{\mathsf{h}_{\phi^{n+1}}(0)\}_{n=0}^{\infty}$ is indeterminate.

3.5. Finite constant valences on generations

In this section we investigate composition operators built on a directed tree with finite constant valences on generations. Let $\mathscr{T} = (V, E)$ be a rootless and leafless directed tree, where V and E stand for the sets of vertices and edges of $\mathscr{T}$, respectively. Denote by $\mathsf{par}(v)$ the parent of $v \in V$. Assume that V is countably infinite. Let μ be a σ-finite measure on 2^V such that $\mu(x) > 0$ for every $x \in V$; call $\mu(x)$ the *mass* of the vertex x. Set $\phi = \mathsf{par}$. By [38, Proposition 2.1.12], there exists a partition $\{G_m\}_{m\in\mathbb{Z}}$ of V such that $G_{m+1} = \bigsqcup_{x\in G_m} \phi^{-1}(\{x\})$ for every $m \in \mathbb{Z}$; call G_m the mth *generation* of $\mathscr{T}$. Assume that $\{\kappa_m\}_{m\in\mathbb{Z}}$ is a two-sided sequence of positive integers and $\{\alpha_m\}_{m\in\mathbb{Z}}$ is a two-sided sequence of positive real numbers such that

$$\phi^{-1}(\{x\}) \text{ has } \kappa_m \text{ elements for all } x \in G_m \text{ and } m \in \mathbb{Z}, \tag{71}$$

$$\mu(x) = \alpha_m \text{ for all } x \in G_m \text{ and } m \in \mathbb{Z}. \tag{72}$$

We call $\{\kappa_m\}_{m\in\mathbb{Z}}$ the *valence sequence* of $\mathscr{T}$. Define $\{\hat{\kappa}_m\}_{m\in\mathbb{Z}} \subseteq (0, \infty)$ by

$$\hat{\kappa}_m = \begin{cases} \prod_{j=0}^{m-1} \kappa_j & \text{if } m \geqslant 1, \\ 1 & \text{if } m = 0, \\ \left(\prod_{j=1}^{-m} \kappa_{-j}\right)^{-1} & \text{if } m \leqslant -1. \end{cases} \tag{73}$$

It is a matter of routine to show that

$$\kappa_m \hat{\kappa}_m = \hat{\kappa}_{m+1}, \quad m \in \mathbb{Z}. \tag{74}$$

Lemma 43. *Under the assumptions above we have*

(i) $\mathsf{h}_{\phi^n}(x) = \dfrac{\alpha_{m+n}}{\alpha_m} \displaystyle\prod_{j=0}^{n-1} \kappa_{m+j}$ *for all* $x \in G_m$, $m \in \mathbb{Z}$ *and* $n \geqslant 1$,

(ii) $\mathsf{E}(\mathsf{h}_{\phi^n}) = \mathsf{h}_{\phi^n}$ *for all* $n \in \mathbb{Z}_+$,

(iii) $\mathcal{D}^\infty(C_\phi)$ *is dense in* $L^2(\mu)$.

Proof. (i) We use induction on n. If $n = 1$, then by (71) and (72), we have

$$\mathsf{h}_\phi(x) = \frac{\mu(\phi^{-1}(\{x\}))}{\alpha_m} = \frac{\alpha_{m+1}\kappa_m}{\alpha_m}, \quad x \in G_m, \, m \in \mathbb{Z}.$$

Now, assume that the induction hypothesis holds for a fixed $n \geqslant 1$. Then

$$
\begin{aligned}
\mathsf{h}_{\phi^{n+1}}(x) &= \frac{\mu(\phi^{-n}(\phi^{-1}(\{x\})))}{\alpha_m} \overset{(72)}{=} \sum_{y\in\phi^{-1}(\{x\})} \frac{\alpha_{m+1}\mu(\phi^{-n}(\{y\}))}{\alpha_m\mu(y)} \\
&= \sum_{y\in\phi^{-1}(\{x\})} \frac{\alpha_{m+1}}{\alpha_m}\mathsf{h}_{\phi^n}(y) = \sum_{y\in\phi^{-1}(\{x\})} \frac{\alpha_{m+1}}{\alpha_m}\frac{\alpha_{m+n+1}}{\alpha_{m+1}}\prod_{j=0}^{n-1}\kappa_{m+j+1} \\
&\overset{(71)}{=} \frac{\alpha_{m+n+1}}{\alpha_m}\kappa_m\prod_{j=1}^{n}\kappa_{m+j} = \frac{\alpha_{m+n+1}}{\alpha_m}\prod_{j=0}^{n}\kappa_{m+j}, \quad x\in G_m,\ m\in\mathbb{Z}.
\end{aligned}
$$

This completes the proof of (i).

(ii) By (i), the function h_{ϕ^n} is constant on $\phi^{-1}(\{x\})$ for all $x \in V$ and $n \geqslant 1$. Since $\mathsf{h}_{\phi^0} \equiv 1$, we get (ii).

(iii) By (i), $\{\chi_{\{x\}}: x \in V\} \subseteq \mathcal{D}^\infty(C_\phi)$, which yields (iii). $\square$

A two-sided sequence $\{a_n\}_{n\in\mathbb{Z}} \subseteq \mathbb{R}_+$ is called a *two-sided Stieltjes moment sequence* if there exists a Borel measure ν on $(0,\infty)$ such that $a_n = \int_{(0,\infty)} s^n \nu(\mathrm{d}s)$ for every $n \in \mathbb{Z}$; the measure ν is called a *representing measure* of $\{a_n\}_{n\in\mathbb{Z}}$. By [4, p. 202], we have

$$
\begin{aligned}
&\{a_n\}_{n\in\mathbb{Z}} \subseteq \mathbb{R}_+ \text{ is a two-sided Stieltjes moment sequence if and only if} \\
&\{a_{n-k}\}_{n=0}^\infty \text{ is a Stieltjes moment sequence for every } k \in \mathbb{Z}_+.
\end{aligned} \tag{75}
$$

Using our main criterion, we provide necessary and sufficient conditions for subnormality of composition operators considered above. To the best of our knowledge, this class of operators is the third one, besides unilateral and bilateral injective weighted shifts (cf. [62, 12]), for which condition (ii) of Theorem 44, known as Lambert's condition (see [40]), characterizes the subnormality in the unbounded case.

Theorem 44. *Under the assumptions of the first paragraph of this section, $\mathcal{D}^\infty(C_\phi)$ is dense in $L^2(\mu)$ and the following four conditions are equivalent:*

(i) *C_ϕ is subnormal,*
(ii) *$\{\|C_\phi^n f\|^2\}_{n=0}^\infty$ is a Stieltjes moment sequence for every $f \in \mathcal{D}^\infty(C_\phi)$,*
(iii) *$\{\mathsf{h}_{\phi^n}(x)\}_{n=0}^\infty$ is a Stieltjes moment sequence for every $x \in V$,*
(iv) *$\{\alpha_m\hat{\kappa}_m\}_{m\in\mathbb{Z}}$ is a two-sided Stieltjes moment sequence (cf. (73)).*

Proof. By Lemma 43(iii), $\overline{\mathcal{D}^\infty(C_\phi)} = L^2(\mu)$. The implications (i)⇒(ii) and (ii)⇒(iii) follow from [11, Proposition 3.2.1] and [13, Theorem 10.4] respectively.

(iii)⇒(iv) Set $\gamma_m = \alpha_m\hat{\kappa}_m$ for $m \in \mathbb{Z}$. An induction argument based on (74) shows that $\hat{\kappa}_{n-m} = \hat{\kappa}_{-m}\prod_{j=0}^{n-1}\kappa_{j-m}$ for all $m \in \mathbb{Z}_+$ and $n \in \mathbb{N}$. Applying Lemma 43(i) implies that $\gamma_{n-m} = \alpha_{-m}\hat{\kappa}_{-m}\mathsf{h}_{\phi^n}(x)$ for every $n \in \mathbb{Z}_+$ and for all $x \in G_{-m}$ and $m \in \mathbb{Z}_+$. This, together with (75), yields (iv).

(iv)⇒(i) Let ν be a representing measure of the two-sided Stieltjes moment sequence $\{\alpha_0^{-1}\alpha_m\hat{\kappa}_m\}_{m\in\mathbb{Z}}$. Define the mapping $P: V \times \mathfrak{B}(\mathbb{R}_+) \to [0,1]$ by

$$P(x,\sigma) = \frac{\alpha_0}{\alpha_m \hat{\kappa}_m} \int\limits_{\sigma} t^m \,\mathrm{d}\nu(t), \quad x \in G_m,\, \sigma \in \mathfrak{B}(\mathbb{R}_+),\, m \in \mathbb{Z}. \tag{76}$$

Since ν is a representing measure of $\{\alpha_0^{-1}\alpha_m\hat{\kappa}_m\}_{m\in\mathbb{Z}}$, we see that P is a family of probability measures. Applying (74), (76) and Lemma 43(i), we deduce that

$$\begin{aligned}\frac{\int_\sigma tP(\phi(x),\,\mathrm{d}t)}{\mathsf{h}_\phi(\phi(x))} &= \frac{\alpha_0}{\alpha_m\kappa_{m-1}\hat{\kappa}_{m-1}} \int\limits_{\sigma} t^m\,\mathrm{d}\nu(t) \\ &= P(x,\sigma), \quad \sigma \in \mathfrak{B}(\mathbb{R}_+),\, x \in G_m,\, m\in\mathbb{Z}.\end{aligned}$$

This means that the family P satisfies (SCC). Since $0 < \mathsf{h}_\phi < \infty$, we infer from Theorem 9 that C_ϕ is subnormal. This completes the proof. $\square$

Remark 45. In view of Theorem 44, C_ϕ is subnormal if and only if there exists a two-sided Stieltjes moment sequence $\{\gamma_m\}_{m\in\mathbb{Z}}$ such that $\alpha_m = \hat{\kappa}_m^{-1}\gamma_m$ for all $m\in\mathbb{Z}$. Hence, if $\mathscr{T}$ is a full κ-ary directed tree, i.e., $\kappa_m = \kappa$ for all $m \in \mathbb{Z}$, then $\hat{\kappa}_m = \kappa^m$ for all $m\in\mathbb{Z}$, and consequently C_ϕ is subnormal if and only if $\{\alpha_m\}_{m\in\mathbb{Z}}$ is a two-sided Stieltjes moment sequence. This characterization of subnormality of C_ϕ does not depend on κ. For $\kappa = 1$, it covers the case of injective bilateral weighted shifts (cf. [35] and [62]). Therefore, a question arises as to whether the composition operator C_ϕ built on a directed tree with the valence sequence $\{\kappa_m\}_{m\in\mathbb{Z}}$ is unitarily equivalent to an orthogonal sum of injective bilateral weighted shifts. The answer is in the negative if $\kappa_m > 1$ for some $m \in \mathbb{Z}$. This is because the adjoint of an injective bilateral weighted shift is injective and $\mathcal{N}(C_\phi^*) \neq \{0\}$. To see that $\mathcal{N}(C_\phi^*) \neq \{0\}$, observe that the linear span of the set $\{\chi_{\{x\}}\colon x \in V\}$ is a core for C_ϕ (use [13, (3.5)] and $\mathsf{h}_\phi < \infty$). Hence $f \in L^2(\mu)$ belongs to $\mathcal{N}(C_\phi^*)$ if and only if $\langle f, \chi_{\phi^{-1}(\{x\})}\rangle = 0$ for every $x \in V$, which implies that for every $x \in \varGamma := \bigcup_{m:\kappa_m>1} G_m$ there exists normalized $h_x \in \chi_{\phi^{-1}(\{x\})}L^2(\mu)$ orthogonal to $\chi_{\phi^{-1}(\{x\})}$ and vanishing on $V\setminus\phi^{-1}(\{x\})$. Then $\{h_x\colon x\in\varGamma\}$ is an orthonormal system in $\mathcal{N}(C_\phi^*)$ and thus $\mathcal{N}(C_\phi^*) \neq \{0\}$. Clearly, if $\varGamma$ is infinite, then $\dim\mathcal{N}(C_\phi^*) = \aleph_0$.

Now we discuss the case of unilateral weighted shifts. By Lemma 43(i), for $\{\kappa_m\}_{m\in\mathbb{Z}} \subseteq \mathbb{N}$ there exists $\{\alpha_m\}_{m\in\mathbb{Z}} \subseteq (0,\infty)$ such that C_ϕ is an isometry. Clearly

$$\mathcal{R}^\infty(C_\phi) := \bigcap_{n=1}^{\infty} \mathcal{R}(C_\phi^n) = \bigcap_{n=1}^{\infty}\bigcap_{x\in V}\Big\{f\in L^2(\mu)\colon f \text{ is constant on } \phi^{-n}(\{x\})\Big\}.$$

Hence, $f \in L^2(\mu)$ belongs to $\mathcal{R}^\infty(C_\phi)$ if and only if f is constant on G_m for every $m\in\mathbb{Z}$. Thus, by (72), $\mathcal{R}^\infty(C_\phi) = \{0\}$ if and only if G_m is infinite for every $m\in\mathbb{Z}$ (by [38, (6.1.3)], the latter is equivalent to $\limsup_{m\to-\infty}\kappa_m \geqslant 2$). If this is the case, then by Wold's decomposition theorem (cf. [23, Theorem 23.7]) C_ϕ is unitarily equivalent to an orthogonal sum of unilateral isometric shifts of multiplicity 1. Otherwise, the unitary part of C_ϕ is nontrivial and so, by Wold's decomposition, C_ϕ is not unitarily equivalent to an orthogonal sum of unilateral weighted shifts.

Regarding Theorem 44, if the masses of vertices of the same generation are not assumed to be constant, then there is no hope to get a characterization of subnormality of C_ϕ. Only a sufficient condition written in terms of consistent systems of probability measures can be provided (cf. [11,12]). The implications (ii)⇒(i) and (iii)⇒(i) are not true in general (cf. [39,13]).

Now we characterize the boundedness and left semi-Fredholmness of subnormal composition operators considered in Theorem 44. For the theory of Fredholmness of general and particular operators, we refer the reader to [33] and [38] respectively.

Proposition 46. *Under the assumptions of the first paragraph of this section, if C_ϕ is subnormal and ν is a representing measure of $\{\alpha_m\hat{\kappa}_m\}_{m\in\mathbb{Z}}$ (cf. (73)), then* $\operatorname{supp}\nu \neq \varnothing$ *and the following assertions hold*:

(i) *C_ϕ is in $\boldsymbol{B}(L^2(\mu))$ if and only if* $\sup(\operatorname{supp}\nu) < \infty$*; moreover, if this is the case, then* $\|C_\phi\|^2 = \sup(\operatorname{supp}\nu)$,

(ii) *if c is a positive real number, then $\|C_\phi f\| \geqslant c\|f\|$ for every $f \in \mathcal{D}(C_\phi)$ if and only if* $\inf(\operatorname{supp}\nu) \geqslant c^2$,

(iii) *C_ϕ is left semi-Fredholm if and only if* $\inf(\operatorname{supp}\nu) > 0$.

Proof. Set $\gamma_m = \alpha_m\hat{\kappa}_m$ for $m \in \mathbb{Z}$. Since $\gamma_0 > 0$, we see that $\operatorname{supp}\nu \neq \varnothing$.

(i) Applying Lemma 2 to the sequences $\{\gamma_{m-k}\}_{m=0}^\infty$, $k \in \mathbb{Z}_+$, we deduce that the two-sided sequence $\big\{\frac{\gamma_{m+1}}{\gamma_m}\big\}_{m\in\mathbb{Z}}$ is monotonically increasing and

$$\sup_{x\in V} \mathsf{h}_\phi(x) \overset{(\dagger)}{=} \sup_{m\in\mathbb{Z}} \frac{\gamma_{m+1}}{\gamma_m} = \sup_{m\in\mathbb{Z}_+} \frac{\gamma_{m+1}}{\gamma_m} = \sup(\operatorname{supp}\nu),$$

where (†) follows from Lemma 43(i) and (74). This and [46, Theorem 1] yields (i).

(ii) We first note that $\{\gamma_{-m}\}_{m\in\mathbb{Z}}$ is a two-sided Stieltjes moment sequence with the representing measure $\nu\circ\tau^{-1}$, where τ is the transformation of $\mathbb{R}_+$ given by $\tau(t) = \frac{1}{t}$ for $t \in (0,\infty)$ and $\tau(0) = 0$. Using the fact that the two-sided sequence $\big\{\frac{\gamma_{m+1}}{\gamma_m}\big\}_{m\in\mathbb{Z}}$ is monotonically increasing (see the previous paragraph) and applying Lemma 2 to the Stieltjes moment sequence $\{\gamma_{-m}\}_{m=0}^\infty$, we get

$$\begin{aligned}\inf\left\{\frac{\gamma_{m+1}}{\gamma_m}\colon m\in\mathbb{Z}\right\} &= \inf\left\{\frac{\gamma_{-m}}{\gamma_{-m-1}}\colon m\in\mathbb{Z}_+\right\} = \frac{1}{\sup\left\{\frac{\gamma_{-(m+1)}}{\gamma_{-m}}\colon m\in\mathbb{Z}_+\right\}} \\ &= \frac{1}{\sup(\operatorname{supp}\nu\circ\tau^{-1})} = \inf(\operatorname{supp}\nu). \end{aligned} \tag{77}$$

By Proposition 4, Lemma 43(i) and (74), $\|C_\phi f\| \geqslant c\|f\|$ for every $f \in \mathcal{D}(C_\phi)$ if and only if $\inf\big\{\frac{\gamma_{m+1}}{\gamma_m}\colon m\in\mathbb{Z}\big\} \geqslant c^2$. This and (77) imply (ii).

(iii) Since C_ϕ is injective closed and densely defined, we infer from the closed graph theorem that C_ϕ is left semi-Fredholm if and only if it is bounded from below. This and (ii) complete the proof. □

Note that under the assumptions of Proposition 46, it may happen that C_ϕ is bounded from below and the measure ν is indeterminate. A sample of such measure appears in the last paragraph of Example 42. In fact, any N-extremal measure on $\mathbb{R}_+$ different from the Krein one meets our requirements (see [39, Section 2.1] for an overview of the theory of indeterminate moment problems).

3.6. Weighted shifts on rootless directed trees

Using Theorem 35, we will show that Theorem 5.1.1 of [11] remains true for weighted shifts on rootless and leafless directed trees with nonzero weights without assuming the density of C^∞-vectors in the underlying ℓ^2-space. Recall that by a *weighted shift* on a rootless directed tree $\mathscr{T} = (V, E)$ with weights $\boldsymbol{\lambda} = \{\lambda_v\}_{v\in V} \subseteq \mathbb{C}$ we mean the operator $S_{\boldsymbol{\lambda}}$ in $\ell^2(V)$ given by

$$\begin{aligned}\mathcal{D}(S_{\boldsymbol{\lambda}}) &= \{f \in \ell^2(V): \varLambda_{\mathscr{T}} f \in \ell^2(V)\},\\ S_{\boldsymbol{\lambda}} f &= \varLambda_{\mathscr{T}} f, \quad f \in \mathcal{D}(S_{\boldsymbol{\lambda}}),\end{aligned}$$

where $\varLambda_{\mathscr{T}}$ is the mapping defined on functions $f: V \to \mathbb{C}$ via

$$(\varLambda_{\mathscr{T}} f)(v) = \lambda_v \cdot f\big(\mathsf{par}(v)\big), \quad v \in V,$$

and $\mathsf{par}(v)$ stands for the parent of v. We refer the reader to [38] for the foundations of the theory of weighted shifts on directed trees.

Theorem 47. *Let $S_{\boldsymbol{\lambda}}$ be a densely defined weighted shift on a rootless and leafless directed tree $\mathscr{T} = (V, E)$ with nonzero weights $\boldsymbol{\lambda} = \{\lambda_v\}_{v\in V}$. Suppose there exists a system $\{\mu_v\}_{v\in V}$ of Borel probability measures on $\mathbb{R}_+$ such that*

$$\mu_u(\sigma) = \sum_{v\in\mathsf{Chi}(u)} |\lambda_v|^2 \int_\sigma \frac{1}{t}\mu_v(\mathrm{d}t), \quad \sigma \in \mathfrak{B}(\mathbb{R}_+),\ u \in V, \tag{78}$$

where $\mathsf{Chi}(u)$ denotes the set of all children of u. Then $S_{\boldsymbol{\lambda}}$ is subnormal.

Proof. In view of [38, Theorem 3.2.1], there is no loss of generality in assuming that all the weights of $S_{\boldsymbol{\lambda}}$ are positive. It follows from [38, Proposition 3.1.10] that V is at most countable. Since $\mathscr{T}$ is rootless, we infer from [38, Proposition 2.1.6] that V is countably infinite. Let $\mathscr{A} = 2^V$ and $\phi(u) = \mathsf{par}(u)$ for $u \in V$. Since $\mathscr{T}$ is rootless and leafless, we see that ϕ is a well-defined surjection. As the weights of $S_{\boldsymbol{\lambda}}$ are positive, we deduce from the proof of [39, Lemma 4.3.1] that there exists a σ-finite measure μ on $\mathscr{A}$ which satisfies the following three conditions:

$$\mu(u) > 0 \text{ for all } u \in V, \tag{79}$$

$$\mu(v) = \lambda_v^2\, \mu(u) \text{ for all } v \in \mathsf{Chi}(u) \text{ and } u \in V, \tag{80}$$

$$S_{\boldsymbol\lambda} \text{ is unitarily equivalent to the composition operator } C_\phi \text{ in } L^2(V, \mathscr{A}, \mu). \tag{81}$$

It follows from (81) that C_ϕ is densely defined, and thus $\mathsf{h}_\phi < \infty$ a.e. $[\mu]$, or equivalently $\mu(\phi^{-1}(\{u\})) < \infty$ for every $u \in V$ (cf. (56)). Since $\mathscr{T}$ is rootless, we infer from (78) that $\mu_u(\{0\}) = 0$ for every $u \in V$. Using (79) and (80), we deduce from (78) that

$$\mu_u(\sigma) = \sum_{v \in \phi^{-1}(\{u\})} \frac{\mu(v)}{\mu(u)} \int\limits_\sigma \frac{1}{t} \mu_v(\mathrm{d}t), \quad \sigma \in \mathfrak{B}(\mathbb{R}_+),\, u \in V,$$

which means that the family $P\colon V \times \mathfrak{B}(\mathbb{R}_+) \to [0,1]$ of probability measures defined by $P(u,\sigma) = \mu_u(\sigma)$ for $u \in V$ and $\sigma \in \mathfrak{B}(\mathbb{R}_+)$ satisfies condition (ii) of Lemma 34. Hence, by applying (79), (81) and Theorem 35, we complete the proof. □

Arguing as in the proof of Theorem 47, one can deduce from Lemma 38 and Theorem 41 that [11, Lemma 4.1.3] and [11, Theorem 5.1.3] remain true for weighted shifts on rootless and leafless directed trees with nonzero weights without assuming the density of C^∞-vectors in the underlying ℓ^2-space.

In the proof of Theorem 47 we have used the fact that a weighted shift on a rootless and leafless directed tree with nonzero weights is unitarily equivalent to a composition operator in an L^2-space. Weighted shifts on directed trees are particular instances of *weighted* composition operators in L^2-spaces. Therefore, one can ask a question whether weighted composition operators in L^2-spaces are unitarily equivalent to composition operators in L^2-spaces. The answer is in the negative regardless of whether the underlying measure space is discrete or not. This can be deduced from Proposition 48 which in turn can be inferred from Proposition B.1.

Proposition 48. *Let $(Y, \mathscr{B}, \nu)$ be a σ-finite measure space, $w\colon Y \to \mathbb{R}$ be a $\mathscr{B}$-measurable function and ψ be the identity transformation of Y. If the weighted composition operator T in $L^2(\nu)$ given by*

$$\mathcal{D}(T) = \big\{f \in L^2(\nu)\colon w \cdot (f \circ \psi) \in L^2(\nu)\big\},$$
$$Tf = w \cdot (f \circ \psi), \quad f \in \mathcal{D}(T),$$

is unitarily equivalent to a composition operator in an L^2-space over a σ-finite measure space, then $|w| = 1$ a.e. $[\nu]$.

Note that Proposition 48 is no longer valid if we allow w to be complex-valued because normal operators are unitarily equivalent to the multiplication operators (cf. [70, Theorem 7.33]; see also [49, Theorem VIII.4]) and there are normal composition operators in L^2-spaces which are not unitary (see e.g., [54, Example 4.2]).

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Appendix A. Composition operators induced by roots of the identity

In Appendix A we will show that a subnormal composition operator induced by an nth root of id_X must be bounded and unitary. The proof depends heavily on the fact that all powers of a composition operator induced by an nth root of id_X are densely defined. We begin by showing that the closures of (*a priori* unbounded) subnormal nth roots of I are unitary. The case of bounded operators can be easily derived from Putnam's inequality (cf. [47, Theorem 1]). Below we present a considerably more elementary proof.

Lemma A.1. *If S is a subnormal operator in a complex Hilbert space $\mathcal{H}$ such that S^n is densely defined and $S^n \subseteq I$ for some integer $n \geqslant 2$, then $\bar{S}$ is unitary.*

Proof. Clearly, S is closable and the closure $\bar{S}$ of S is subnormal. By [57, Proposition 5.3], $\bar{S}^n$ is closed. Since S^n is densely defined, we deduce that $\bar{S}^n = I$. Hence, by the closed graph theorem, $\bar{S} \in \boldsymbol{B}(\mathcal{H})$. Let $N \in \boldsymbol{B}(\mathcal{K})$ be a minimal normal extension of $\bar{S}$ acting in a complex Hilbert space $\mathcal{K}$. By minimality of N, $N^n = I_{\mathcal{K}}$. This implies that $|N|^{2n} = I_{\mathcal{K}}$, and so $|N| = I_{\mathcal{K}}$. Therefore, N is unitary and consequently $\bar{S}$ is an isometry which is onto (because $\bar{S}^n = I$). □

Lemma A.1 is no longer true if we do not assume S^n to be densely defined. Indeed, for every integer $n \geqslant 2$, there exists an unbounded closed symmetric operator[7] S such that S^{n-1} is densely defined and $\mathcal{D}(S^n) = \{0\}$ (cf. [51, Remark 4.6.3]; see also [44,19] for $n = 2$). Then $S^n \subseteq I$, but S is not a normal operator.

In the rest of Appendix A we assume that $(X, \mathscr{A}, \mu)$ is a σ-finite measure space. A transformation ϕ of X is called *$\mathscr{A}$-bimeasurable* if $\phi(\Delta) \in \mathscr{A}$ and $\phi^{-1}(\Delta) \in \mathscr{A}$ for every $\Delta \in \mathscr{A}$. The following lemma is inspired by [16, Proposition 4.1(vi)].

[7] Recall that symmetric operators are always subnormal (cf. [1, Theorem 1 in Appendix I.2]).

Lemma A.2. *If $\{\phi_j\}_{j=1}^n$ is a finite sequence of bijective $\mathscr{A}$-bimeasurable nonsingular transformations of X such that $\phi_1 \circ \cdots \circ \phi_n = \mathrm{id}_X$ and $n \geqslant 2$, then*

$$\mathsf{h}_{\phi_1} \cdot \mathsf{h}_{\phi_2} \circ \phi_1^{-1} \cdots \mathsf{h}_{\phi_n} \circ (\phi_1 \circ \cdots \circ \phi_{n-1})^{-1} = 1 \text{ a.e. } [\mu]. \tag{A.1}$$

Proof. Applying the measure transport theorem repeatedly and an induction argument, we get

$$\begin{aligned}
\mu(\varDelta) &= \mu(\phi_n^{-1}((\phi_1 \circ \cdots \circ \phi_{n-1})^{-1}(\varDelta))) \\
&= \int_X \chi_\varDelta \circ \phi_1 \circ \cdots \circ \phi_{n-1} \cdot \mathsf{h}_{\phi_n} \circ \phi_{n-1}^{-1} \circ \phi_{n-1} \, \mathrm{d}\mu \\
&= \int_X \chi_\varDelta \circ \phi_1 \circ \cdots \circ \phi_{n-2} \cdot \mathsf{h}_{\phi_{n-1}} \cdot \mathsf{h}_{\phi_n} \circ \phi_{n-1}^{-1} \, \mathrm{d}\mu \\
&= \int_X \chi_\varDelta \circ \phi_1 \circ \cdots \circ \phi_{n-2} \cdot \mathsf{h}_{\phi_{n-1}} \circ \phi_{n-2}^{-1} \circ \phi_{n-2} \cdot \mathsf{h}_{\phi_n} \circ \phi_{n-1}^{-1} \circ \phi_{n-2}^{-1} \circ \phi_{n-2} \, \mathrm{d}\mu \\
&= \int_X \chi_\varDelta \circ \phi_1 \circ \cdots \circ \phi_{n-3} \cdot \mathsf{h}_{\phi_{n-2}} \cdot \mathsf{h}_{\phi_{n-1}} \circ \phi_{n-2}^{-1} \cdot \mathsf{h}_{\phi_n} \circ \phi_{n-1}^{-1} \circ \phi_{n-2}^{-1} \, \mathrm{d}\mu \\
&\ \vdots \\
&= \int_X \chi_\varDelta \cdot \mathsf{h}_{\phi_1} \cdot \mathsf{h}_{\phi_2} \circ \phi_1^{-1} \cdots \mathsf{h}_{\phi_n} \circ (\phi_1 \circ \cdots \circ \phi_{n-1})^{-1} \, \mathrm{d}\mu, \quad \varDelta \in \mathscr{A}.
\end{aligned}$$

By the σ-finiteness of μ, this implies (A.1). $\square$

We are now ready to prove the main result of Appendix A.

Proposition A.3. *If ϕ is a nonsingular transformation of X such that $\phi^n = \mathrm{id}_X$ for some integer $n \geqslant 2$, then the following conditions hold:*

(i) *ϕ^m is a bijective and nonsingular transformation of X for every $m \in \mathbb{Z}$,*
(ii) *$\mathcal{D}(C_\phi^m) = \mathcal{D}(C_\phi^{n-1})$ for every integer $m \geqslant n$,*
(iii) *$C_\phi^m = C_\phi^r|_{\mathcal{D}^\infty(C_\phi)}$ for all $m, r \in \mathbb{Z}_+$ such that $m \geqslant n$ and $r \equiv m$ (mod n),*
(iv) *$\mathcal{D}^\infty(C_\phi)$ is a core for C_ϕ^m for every $m \in \mathbb{Z}_+$,*
(v) *$C_\phi \in \boldsymbol{B}(L^2(\mu))$ if and only if C_ϕ^n is closed,*
(vi) *C_ϕ is subnormal if and only if C_ϕ is unitary.*

Proof. (i) Since $\phi^n = \mathrm{id}_X$, the transformation ϕ is bijective and $\phi^{-1} = \phi^{n-1}$. This implies that ϕ is $\mathscr{A}$-bimeasurable and ϕ^{-1} is nonsingular. Hence (i) is satisfied.

(ii) and (iii) follow from [64, Proposition 14] and the equality $\phi^n = \mathrm{id}_X$.

(iv) If $j \in \{1, \ldots, n\}$, then by Lemma A.2, applied to $n = 2$, $\phi_1 = \phi^j$ and $\phi_2 = \phi^{n-j}$, we deduce that $\mathsf{h}_{\phi^j} < \infty$ a.e. $[\mu]$. In view of [13, Corollary 4.5], this implies that C_ϕ^n is densely defined. Hence, by (ii), $\mathcal{D}^\infty(C_\phi)$ is dense in $L^2(\mu)$. Applying [13, Theorem 4.7] completes the proof of (iv).

(v) Suppose C_ϕ^n is closed. Since $C_\phi^n \subseteq I$, we infer from (iv) that $C_\phi^n = I$ and so $\mathcal{D}(C_\phi) = L^2(\mu)$. Hence, by the closed graph theorem, $C_\phi \in \boldsymbol{B}(L^2(\mu))$. The reverse implication is obvious.

(vi) This condition follows from (iv) and Lemma A.1. $\square$

Example A.4. We will show that for every integer $n \geqslant 3$, there exists a nonsingular transformation ϕ of a discrete measure space $(X, \mathscr{A}, \mu)$ such that $\phi^n = \mathrm{id}_X$ and $\mathcal{D}(C_\phi^{n-1}) \subsetneq \mathcal{D}(C_\phi^{n-2}) \subsetneq \ldots \subsetneq \mathcal{D}(C_\phi)$. By [64, Proposition 14], it suffices to show that $\mathcal{D}(C_\phi^{n-1}) \subsetneq \mathcal{D}(C_\phi^{n-2})$. Set $X = \mathbb{Z}_+$ and $\mathscr{A} = 2^X$, and take a sequence $\{\gamma_k\}_{k=0}^\infty \subset (0, \infty)$ tending to ∞. Define μ by $\mu(j + kn) = \gamma_k^j$ for $j \in \{0, \ldots, n-1\}$ and $k \in \mathbb{Z}_+$. Let ϕ be the transformation of X given by $\phi(j + kn) = \widehat{j+1} + kn$ for $j \in \{0, \ldots, n-1\}$ and $k \in \mathbb{Z}_+$, where $\widehat{j+1} = j + 1$ if $j + 1 < n$ and $\widehat{j+1} = 0$ if $j + 1 = n$. It is clear that $\phi^n = \mathrm{id}_X$. Suppose that, contrary to our claim, $\mathcal{D}(C_\phi^{n-1}) = \mathcal{D}(C_\phi^{n-2})$. Then, by [13, Proposition 4.3], there exists $c \in (0, \infty)$ such that $\mathsf{h}_{\phi^{n-1}} \leqslant c(1 + \sum_{l=1}^{n-2} \mathsf{h}_{\phi^l})$. Since $\mathsf{h}_{\phi^{n-1}}(n - 2 + kn) = \gamma_k$ and $\mathsf{h}_{\phi^l}(n - 2 + kn) = \gamma_k^{-l}$ for all $l \in \{1, \ldots, n-2\}$ and $k \in \mathbb{Z}_+$, we arrive at the contradiction.

Appendix B. Symmetric composition operators

We will show that symmetric composition operators are selfadjoint and unitary.

Proposition B.1. *Let $(X, \mathscr{A}, \mu)$ be a σ-finite measure space and ϕ be a nonsingular transformation of X. If C_ϕ is symmetric, then C_ϕ is selfadjoint and unitary, and $C_\phi^2 = I$. If C_ϕ is positive and symmetric, then $C_\phi = I$.*

Proof. Since symmetric operators are formally normal, we infer from [13, Theorem 9.4] that if C_ϕ is symmetric, then C_ϕ is normal and consequently selfadjoint. For clarity, the rest of the proof will be divided into two steps.

Step 1. If C_ϕ is positive and selfadjoint, then $C_\phi = I$.

Indeed, by [13, Proposition 6.2], C_ϕ is injective. Since $C_\phi = |C_\phi|$, the partial isometry U in the polar decomposition of C_ϕ is the identity operator on $L^2(\mu)$. This together with [13, Proposition 7.1(iv)] yields

$$f \circ \phi = f \cdot \sqrt{\mathsf{h}_\phi \circ \phi} \text{ a.e. } [\mu], \quad f \in L^2(\mu). \tag{B.1}$$

Take $\varDelta \in \mathscr{A}$ such that $\mu(\varDelta) < \infty$. Substituting $f = \chi_\varDelta$ into (B.1) and using (5), we see that $\mu(\varDelta \setminus \phi^{-1}(\varDelta)) = \mu(\phi^{-1}(\varDelta) \setminus \varDelta) = 0$ and thus $\mu(\varDelta) = (\mu \circ \phi^{-1})(\varDelta)$. Since μ is σ-finite, we conclude that $\mu = \mu \circ \phi^{-1}$. Therefore $\mathsf{h}_\phi = 1$ a.e. $[\mu]$.

By [13, Proposition 7.1(i)], $C_\phi = |C_\phi|$ is the operator of multiplication by $\mathsf{h}_\phi^{1/2}$ and thus $C_\phi = I$.

Step 2. If C_ϕ is selfadjoint, then C_ϕ is unitary and $C_\phi^2 = I$.

Indeed, by [70, Theorem 7.19], C_ϕ^2 is selfadjoint. Hence C_ϕ^2 is closed. By [13, Corollary 4.2] (with $n = 2$), we have

$$C_\phi^* C_\phi = C_\phi C_\phi^* = C_\phi^2 = \overline{C_\phi^2} = C_{\phi^2}, \tag{B.2}$$

which means that C_{ϕ^2} is positive and selfadjoint. It follows from Step 1 that $C_{\phi^2} = I$. Therefore, by (B.2), C_ϕ is unitary (see also Lemma A.1) and $C_\phi^2 = I$.

Putting this all together completes the proof. □

Adapting [15, Example 3.2] to the present context, one can show that the equality $C_\phi = I$ does not imply that $\phi = \mathrm{id}_X$ a.e. $[\mu]$. It may even happen that the set $\{x \in X\colon \phi(x) = x\}$ is not $\mathscr{A}$-measurable.

Example B.2. We will show that there exists a selfadjoint composition operator which is not positive. Set $X = \mathbb{Z}_+$ and $\mathscr{A} = 2^X$. Consider a measure μ on $\mathscr{A}$ such that $0 < \mu(2k) = \mu(2k+1) < \infty$ for all $k \in \mathbb{Z}_+$, and the transformation ϕ of X given by $\phi(2k) = 2k+1$ and $\phi(2k+1) = 2k$ for $k \in \mathbb{Z}_+$. Then $\phi^2 = \mathrm{id}_X$ and consequently $\phi^{-1} = \phi$. It is clear that $\mathsf{h}_\phi = 1$ and thus $C_\phi \in \boldsymbol{B}(L^2(\mu))$. Since $\phi^{-1}(\mathscr{A}) = \mathscr{A}$, we deduce from [13, Corollary 7.3 and Remark 7.4] that $C_\phi^* f = \mathsf{h}_\phi \cdot f \circ \phi^{-1} = f \circ \phi = C_\phi f$ for all $f \in L^2(\mu)$. Hence C_ϕ is selfadjoint. Since $C_\phi f = -f$, where $f(l) = (-1)^l \chi_{\{0,1\}}(l)$ for $l \in \mathbb{Z}_+$, the operator C_ϕ is not positive.

Appendix C. Orthogonal sums of composition operators

Let $(X, \mathscr{A}, \mu)$ be a σ-finite measure space and ϕ be a nonsingular transformation of X. Define $\mathscr{A}(\phi) = \{Y \in \mathscr{A}\colon \phi(Y) \subseteq Y \text{ and } \phi(X \setminus Y) \subseteq X \setminus Y\}$. Since $\mathscr{A}(\phi) = \{Y \in \mathscr{A}\colon \phi^{-1}(Y) = Y\}$, $\mathscr{A}(\phi)$ is a σ-algebra. For nonempty $Y \in \mathscr{A}(\phi)$, we set $\mathscr{A}_Y = \{\Delta \in \mathscr{A}\colon \Delta \subseteq Y\}$, $\mu_Y = \mu|_{\mathscr{A}_Y}$ and $\phi_Y = \phi|_Y$. Clearly, $(Y, \mathscr{A}_Y, \mu_Y)$ is a σ-finite measure space and ϕ_Y is a nonsingular transformation of Y. Given $N \in \mathbb{N} \cup \{\infty\}$, we write J_N for the set of all integers n such that $1 \leqslant n \leqslant N$.

Proposition C.1. *Suppose $N \in \mathbb{N} \cup \{\infty\}$ and $\{Y_n\}_{n=1}^N \subseteq \mathscr{A}(\phi)$ is a sequence of pairwise disjoint nonempty sets. Set $Y = \bigcup_{n=1}^N Y_n$. Then the following hold:*

(i) *$\chi_{Y_n} L^2(\mu)$ reduces C_ϕ and $C_\phi|_{\chi_{Y_n} L^2(\mu)}$ is unitarily equivalent to $C_{\phi_{Y_n}}$ for every $n \in J_N$,*

(ii) *$C_\phi|_{\chi_Y L^2(\mu)} = \bigoplus_{n=1}^N C_\phi|_{\chi_{Y_n} L^2(\mu)}$,*

(iii) *$C_\phi|_{\chi_Y L^2(\mu)}$ is unitarily equivalent to $\bigoplus_{n=1}^N C_{\phi_{Y_n}}$.*

Proof. Since the orthogonal projection P_{Y_n} of $L^2(\mu)$ onto $\chi_{Y_n} L^2(\mu)$ is given by $P_{Y_n}(f) = \chi_{Y_n} \cdot f$ for $f \in L^2(\mu)$, we see that $(P_{Y_n} f) \circ \phi = P_{Y_n}(f \circ \phi)$ for all $f \in \mathcal{D}(C_\phi)$. Hence $P_{Y_n} C_\phi \subseteq C_\phi P_{Y_n}$. The rest of the proof of (i) is straightforward. Since $\chi_Y L^2(\mu) = \bigoplus_{n=1}^N \chi_{Y_n} L^2(\mu)$, (ii) follows from (i) and the fact that C_ϕ is closed. Finally, (iii) is a direct consequence of (i) and (ii). □

Corollary C.2. *An orthogonal sum of countably many composition operators in L^2-spaces is unitarily equivalent to a composition operator in an L^2-space.*

Proof. Let $\{(X_n, \mathscr{A}_n, \mu_n)\}_{n=1}^N$ be a sequence of σ-finite measure spaces and $\{\phi_n\}_{n=1}^N$ be a sequence of nonsingular transformations ϕ_n of X_n, where $N \in \mathbb{N} \cup \{\infty\}$. Set $X = \bigcup_{n=1}^N X_n \times \{n\}$, $\mathscr{A} = \big\{ \bigcup_{n=1}^N \varDelta_n \times \{n\} \colon \varDelta_n \in \mathscr{A}_n \ \forall n \in J_N \big\}$ and $\mu(\varDelta) = \sum_{n=1}^N \mu_n(\varDelta_n)$ for $\varDelta = \bigcup_{n=1}^N \varDelta_n \times \{n\}$ ($\varDelta_n \in \mathscr{A}_n$). Define the transformation ϕ of X by $\phi((x,n)) = (\phi_n(x), n)$ for $x \in X_n$ and $n \in J_N$. Then $(X, \mathscr{A}, \mu)$ is a σ-finite measure space and ϕ is nonsingular. Applying Proposition C.1 to $Y_n := X_n \times \{n\}$, we deduce that $\bigoplus_{n=1}^N C_{\phi_n}$ is unitarily equivalent to C_ϕ. □

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